Quantum Group and the Hofstadter Problem in Graphene

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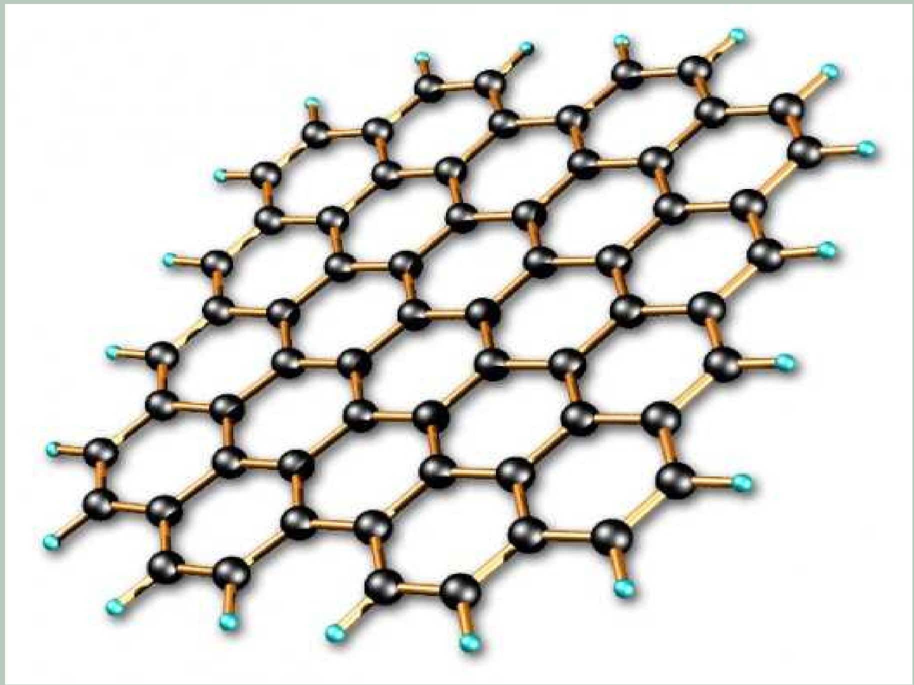
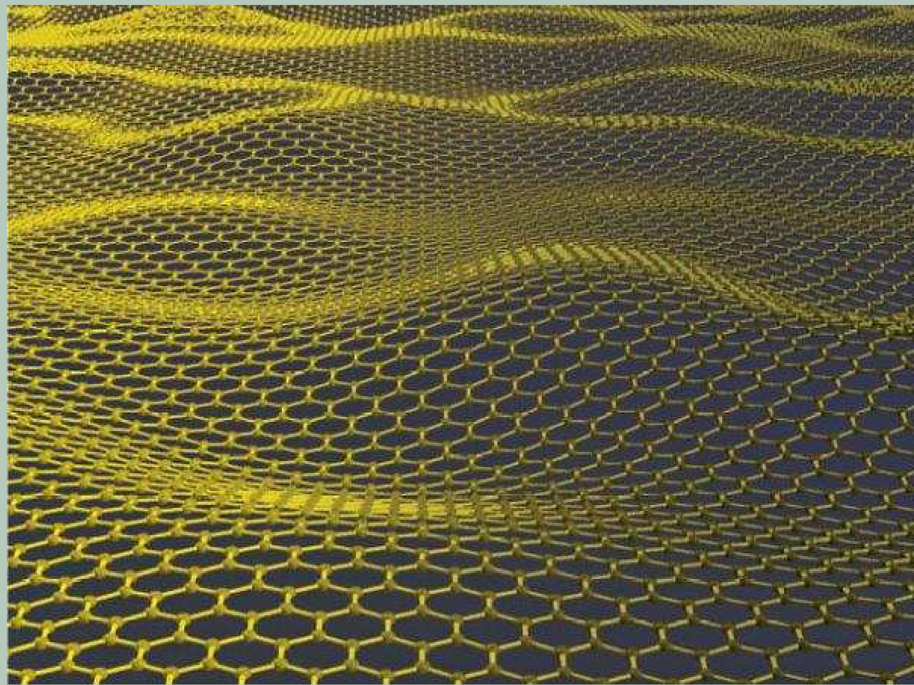
Graphene



Ripples in Graphene

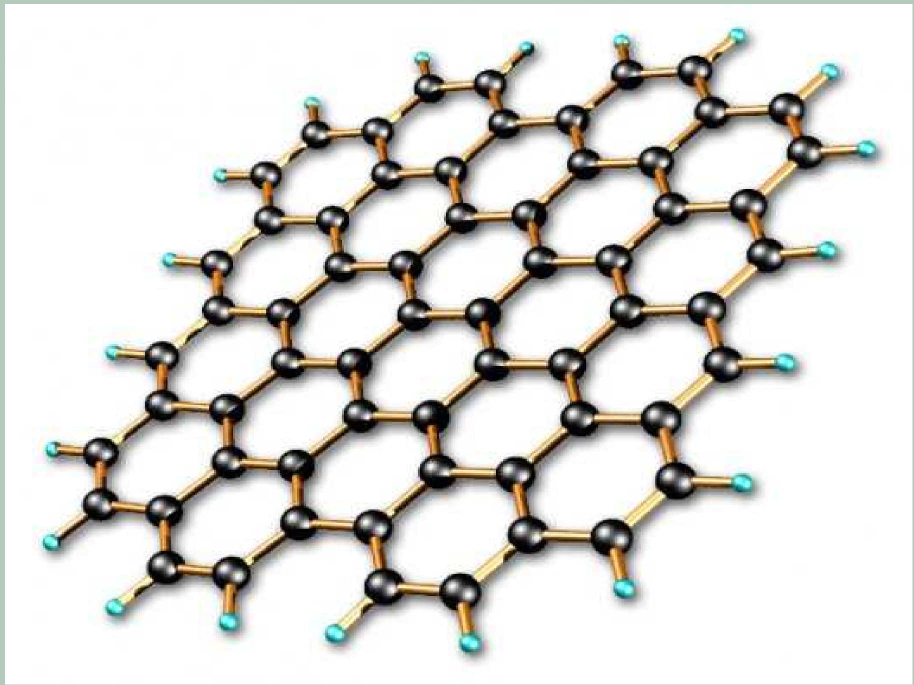


"Ripples arise then due to spontaneous symmetry breaking, following a mechanism similar to that responsible for the condensation of the Higgs field in relativistic field theories"

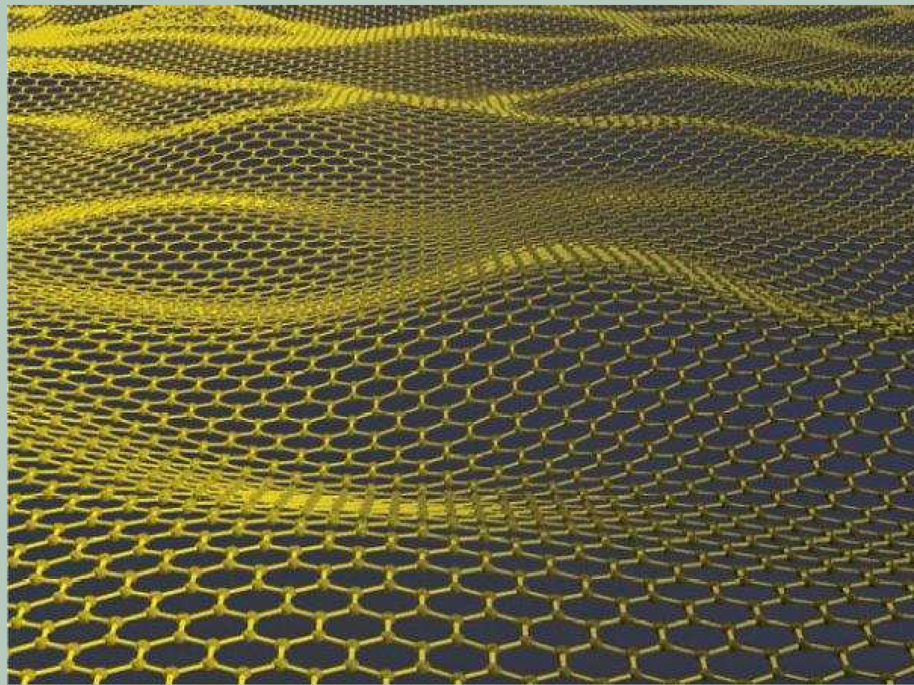
P. San-Jose, J. González, F. Guinea, PRL 106 (2011)

Quantum Group and the Hofstadter Problem in Graphene

Graphene



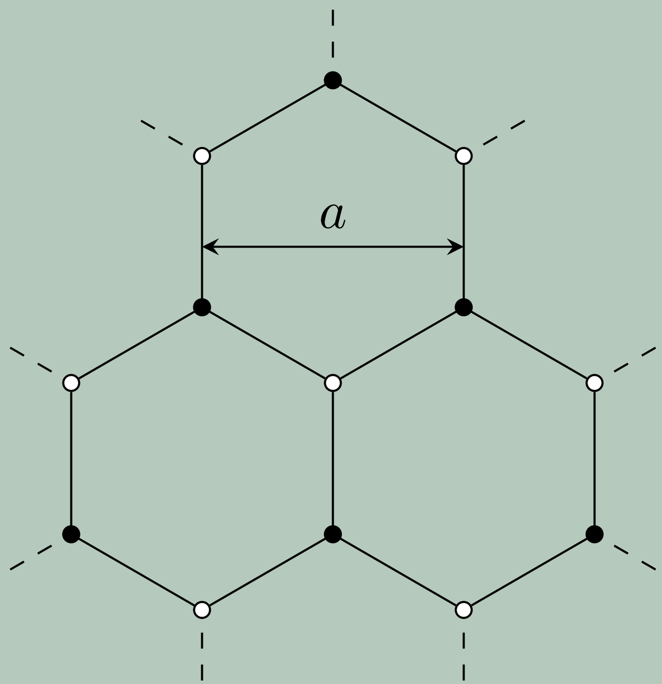
Ripples in Graphene



- Connection to The Quantum Group $U_q(sl_2)$
- Reducibility of Characteristic Polynomials
- Connection to SUSY Quantum Mechanics

Electrons on a Honeycomb Lattice

Tight-binding model with the nearest neighbouring hoppings of electrons



Lattice site coordinate $\boldsymbol{r}_{n_1 n_2}$, shortly $\boldsymbol{r}_n$

$$H = \sum_{mn} \left[f_{\bullet}^{\dagger}(\boldsymbol{r}_m) f_{\circ}(\boldsymbol{r}_n) + f_{\circ}^{\dagger}(\boldsymbol{r}_n) f_{\bullet}(\boldsymbol{r}_m) \right]$$

$$H = \int \epsilon(\boldsymbol{k}) \left[f_{+}^{\dagger}(\boldsymbol{k}) f_{+}(\boldsymbol{k}) - f_{-}^{\dagger}(\boldsymbol{k}) f_{-}(\boldsymbol{k}) \right] d\boldsymbol{k}$$

$$\epsilon(\boldsymbol{k}) = \sqrt{1 + 4\cos\left(\tfrac{1}{2}\, k_x a\right) \cos\left(\tfrac{\sqrt{3}}{2}\, k_y a\right) + 4\cos^2\left(\tfrac{1}{2}\, k_x a\right)}$$

Electrons on a Honeycomb Lattice

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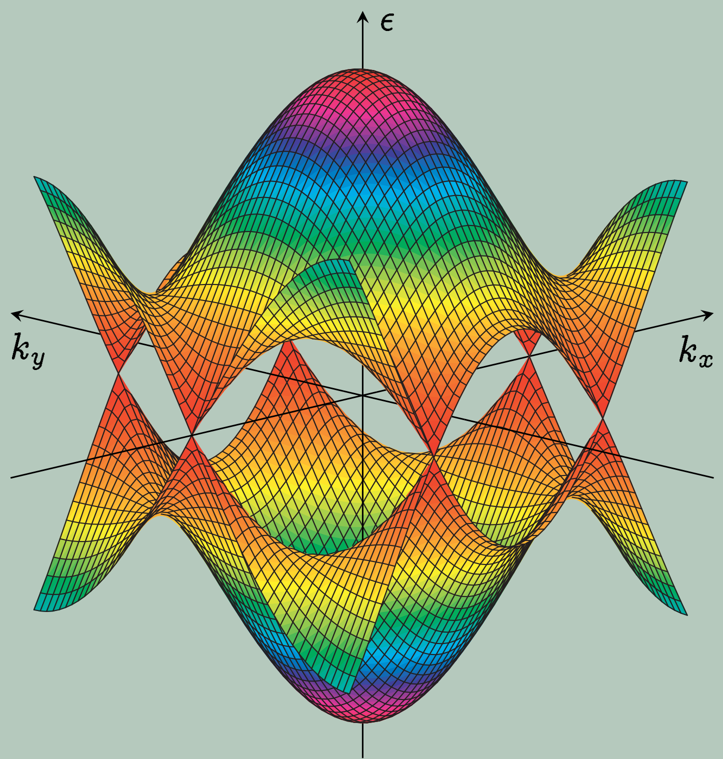
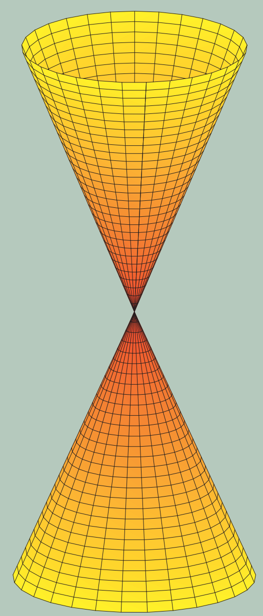


Low-energy excitations → Massless Dirac



$$\epsilon(\Delta\boldsymbol{k}) = \pm|\Delta\boldsymbol{k}|$$

$$L = i\psi\gamma_n\partial_n\psi$$

$$L = \psi\gamma_n(i\partial_n + A_n)\psi$$

Electrons on a Honeycomb Lattice in Magnetic Field

"Peierls Substitution" $f_\bullet^\dagger(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n) \;\to\; e^{-i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f_\bullet^\dagger(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n)$

$$\theta(\boldsymbol{r}_m|\boldsymbol{r}_n) = \frac{e}{\hbar}\int_{\boldsymbol{r}_n}^{\boldsymbol{r}_m} \boldsymbol{A} d\boldsymbol{l} \qquad B = \mathrm{rot}\,\boldsymbol{A}$$

Gauge Invariance

$$\begin{cases} A_j(\boldsymbol{r}) \;\to\; A_j(\boldsymbol{r}) + \partial_j \lambda(\boldsymbol{r}) \\ f(\boldsymbol{r}_m) \;\to\; e^{-i\lambda(\boldsymbol{r}_m)} f(\boldsymbol{r}_m) \end{cases}$$

$$e^{-i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f_\bullet^\dagger(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n) = \mathrm{inv}$$

$$H = \sum_{mn}\left[e^{-i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f_\bullet^\dagger(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n) + e^{+i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f_\circ^\dagger(\boldsymbol{r}_n) f_\bullet(\boldsymbol{r}_m)\right]$$

Electrons on a Honeycomb Lattice in Magnetic Field

$$H = \sum_{mn} \left[e^{-i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f^\dagger_\bullet(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n) + e^{+i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f^\dagger_\circ(\boldsymbol{r}_n) f_\bullet(\boldsymbol{r}_m) \right]$$

$$\Phi = \frac{\nu}{N}\Phi_0 \qquad \Phi = B \cdot Area_{\hexagon} \qquad \Phi_0 = \frac{2\pi\hbar}{e}$$

System splits into $(2N \times 2N)$-dimensional independent blocks $\mathcal{H}(\boldsymbol{k})$

$$H = \int \Psi^\dagger(\boldsymbol{k})\, \mathcal{H}(\boldsymbol{k})\, \Psi(\boldsymbol{k})\, d\boldsymbol{k}$$

Hofstadter Problem: how the flux $\frac{\nu}{N}$ affects the spectrum of $\mathcal{H}(\boldsymbol{k})$?

Eigenvalue Problem

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} 0 & \Delta^\dagger(\boldsymbol{k}) \\ \Delta(\boldsymbol{k}) & 0 \end{pmatrix}_{2N\times 2N}$$

$$\Delta = \mathbb{I} + e^{-ik_x a}\beta^\dagger\Lambda + e^{-ik_y a}\Lambda\beta$$

$$\Lambda = \mathrm{diag}(q^1, q^2 \ldots, q^N)$$

$$q = e^{i\pi(\nu/N)}$$

$$\beta = \begin{pmatrix} 0 & \mathbf{1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Eigenvalue Problem

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Eigenvalue Equation

$$\begin{pmatrix} 0 & \Delta^\dagger(\boldsymbol{k}) \\ \Delta(\boldsymbol{k}) & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \epsilon(\boldsymbol{k})\begin{pmatrix} \xi \\ \zeta \end{pmatrix}$$

$$\begin{cases} \Delta\xi = \epsilon\zeta \\ \Delta^\dagger\zeta = \epsilon\xi \end{cases} \Rightarrow \begin{cases} \Delta^\dagger\Delta\xi = \epsilon^2\xi \\ \Delta\Delta^\dagger\zeta = \epsilon^2\zeta \end{cases}$$

Characteristic Polynomial

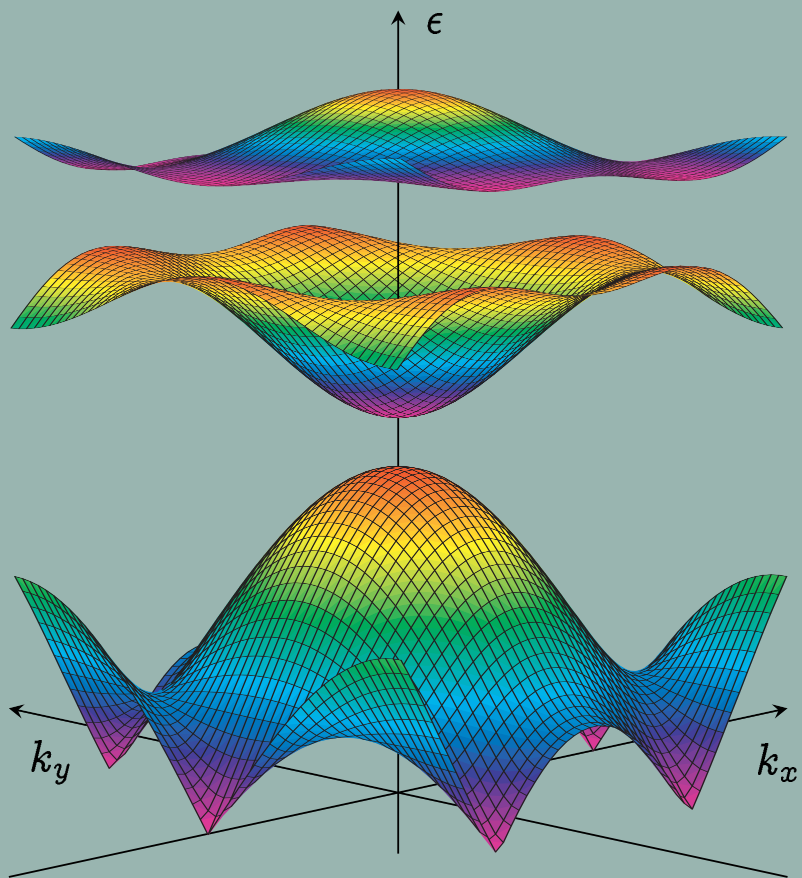
$$\Omega_N^\nu(\epsilon^2) \equiv \det(\Delta^\dagger\Delta - \mathbb{I}\cdot\epsilon^2)$$

$$= \det(\Delta\Delta^\dagger - \mathbb{I}\cdot\epsilon^2)$$

One-Particle Spectrum

$N = 3 \qquad \Omega_3^\nu(\epsilon^2) = 0$

$$\Omega_3^\nu(z) = z^3 - 9z^2 + 18z - \omega_0(\boldsymbol{k})$$



General N

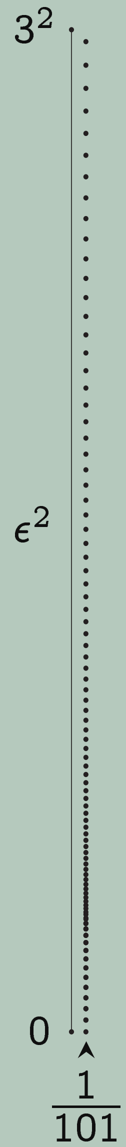
- N bands $\epsilon_1^2(\boldsymbol{k}), \ldots, \epsilon_N^2(\boldsymbol{k})$
- Bands show no touch points
  $\epsilon_1^2(\boldsymbol{k}) < \epsilon_2^2(\boldsymbol{k}) < \cdots < \epsilon_N^2(\boldsymbol{k})$
- Spectrum bound $\epsilon_N^2(\boldsymbol{k}) \leqslant 3^2$
- Bands flatten as N increases
  $\epsilon_1^2 < \epsilon_2^2 < \cdots < \epsilon_N^2$
- Zero modes at special $\boldsymbol{k}$'s

Hofstadter Butterfly

$$\texttt{det}\left(\Delta^\dagger \Delta - \mathbb{I} \cdot \epsilon^2\right) = 0$$

$$\frac{\nu}{N} = \frac{1 \div 100}{101}$$

Hofstadter Butterfly



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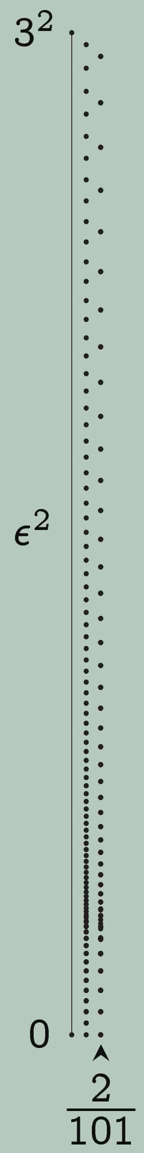

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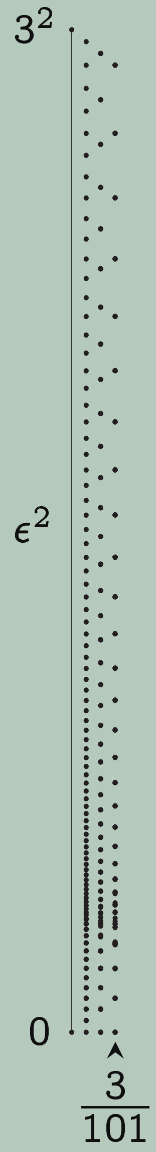

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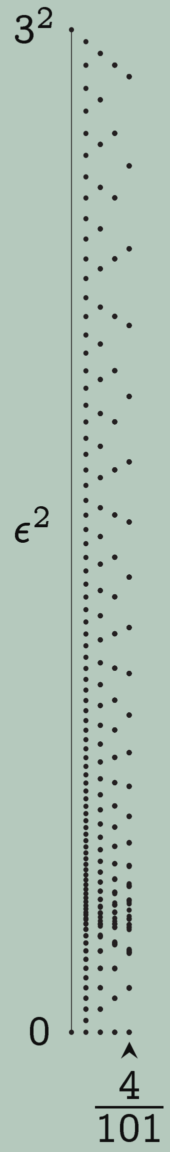

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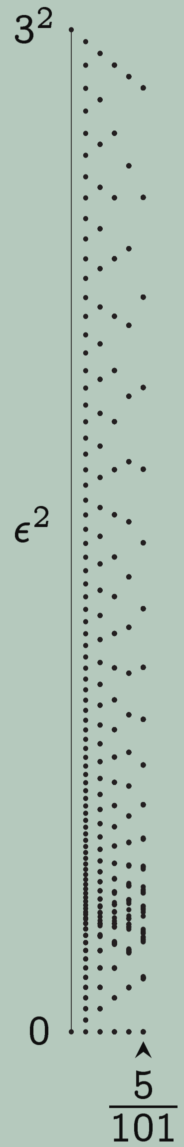

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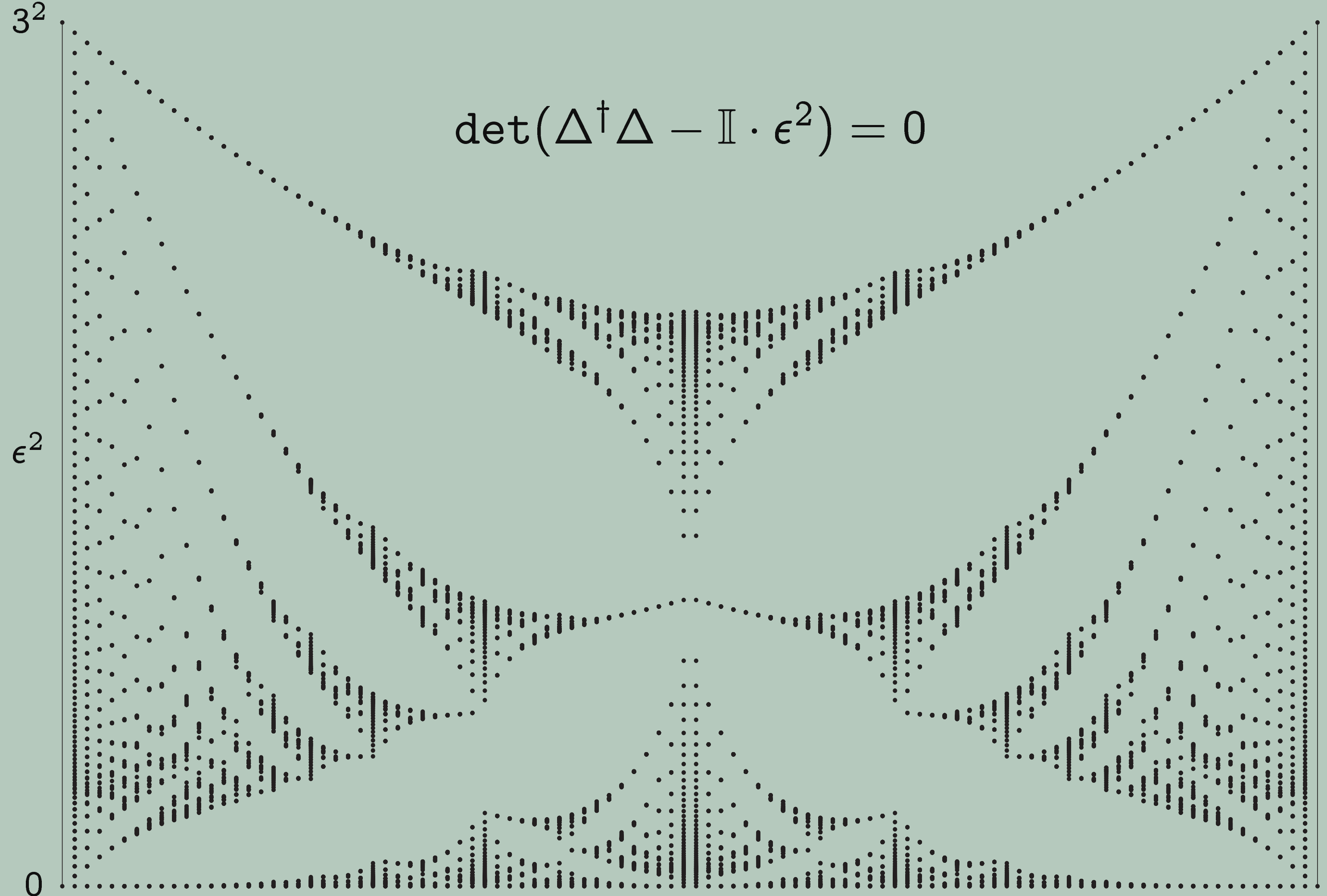

$$\frac{\nu}{N} = \frac{1 \div 100}{101}$$

"Butterfly" discovered by Hofstadter (1976) for square lattice

The Quantum Group $U_q(sl_2)$

$$\mathcal{H} = \begin{pmatrix} 0 & \Delta^\dagger \\ \Delta & 0 \end{pmatrix}$$

$$\Delta = \mathbb{I} + e^{-ik_x a}\beta^\dagger \Lambda + e^{-ik_y a}\Lambda\beta$$

$$\Lambda = \mathrm{diag}(q^1, q^2 \ldots, q^N)$$

$$\beta = \begin{pmatrix} 0 & \mathbf{1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

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$$E = \frac{e^{-ik_x a}\beta^\dagger\Lambda + e^{-ik_y a}\Lambda\beta}{i(q - q^{-1})}$$

$$F = \frac{e^{+ik_x a}\Lambda^\dagger\beta + e^{+ik_y a}\beta^\dagger\Lambda^\dagger}{i(q - q^{-1})}$$

The Quantum Group $U_q(sl_2)$

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The Quantum Group $U_q(sl_2)$

$$\mathcal{H} = \begin{pmatrix} 0 & \mathbb{I} + i(q - q^{-1})F \\ \mathbb{I} + i(q - q^{-1})E & 0 \end{pmatrix}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$E = \frac{e^{-ik_x a}\beta^\dagger \Lambda + e^{-ik_y a}\Lambda\beta}{i(q - q^{-1})}$$

$$KEK^{-1} = q^2 E$$

$$F = \frac{e^{+ik_x a}\Lambda^\dagger \beta + e^{+ik_y a}\beta^\dagger\Lambda^\dagger}{i(q - q^{-1})}$$

$$KFK^{-1} = q^{-2} F$$

$$K = q e^{+i(k_x - k_y)a}\Lambda\beta\Lambda^\dagger\beta$$

The Quantum Group $U_q(sl_2)$

$$\mathcal{H} = \begin{pmatrix} 0 & \mathbb{I} + i(q - q^{-1})F \\ \mathbb{I} + i(q - q^{-1})E & 0 \end{pmatrix}$$

Quantum Group $U_q(sl_2)$

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The Quantum Group $U_q(sl_2)$

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Quantum Group $U_q(sl_2)$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$KEK^{-1} = q^2 E$$

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Cyclic Representation of $U_q(sl_2)$

$$E = \frac{e^{-ik_x a}\beta^\dagger \Lambda + e^{-ik_y a}\Lambda\beta}{i(q - q^{-1})}$$

$$F = \frac{e^{+ik_x a}\Lambda^\dagger \beta + e^{+ik_y a}\beta^\dagger \Lambda^\dagger}{i(q - q^{-1})}$$

$$K = q e^{+i(k_x - k_y)a}\Lambda\beta\Lambda^\dagger\beta$$

The Quantum Group $U_q(sl_2)$

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Quantum Group $U_q(sl_2)$

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$$KEK^{-1} = q^2 E$$

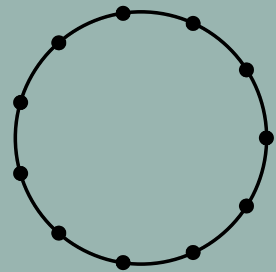
$$KFK^{-1} = q^{-2} F$$

Cyclic Representation of $U_q(sl_2)$

(only for $q^N = \pm 1$)

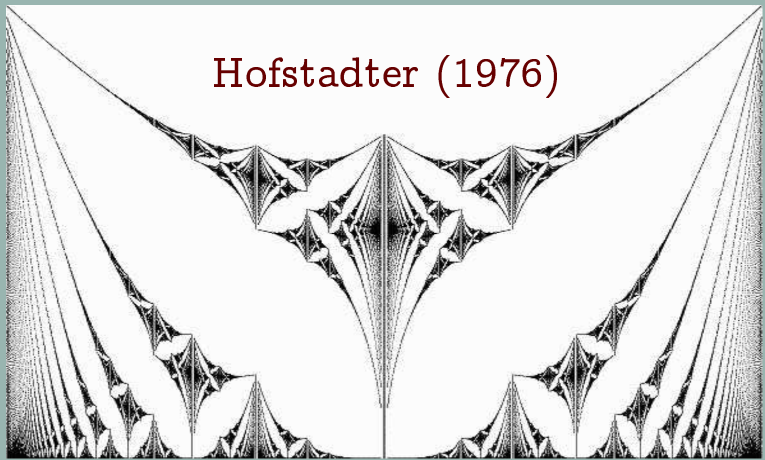
$$E : \psi_n \to \psi_{n+1}$$

$$F : \psi_n \to \psi_{n-1}$$



No highest/lowest weight vectors

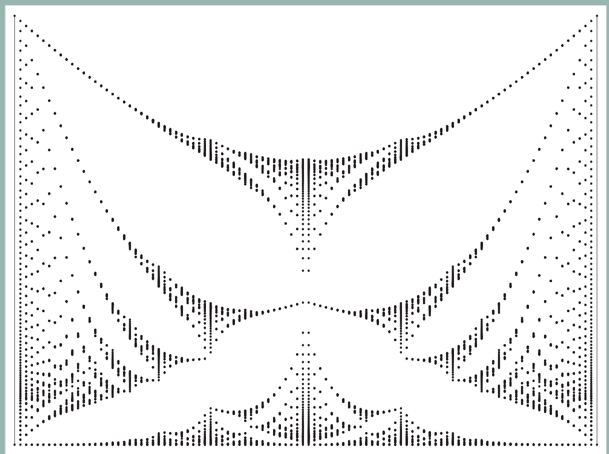
The Quantum Group $U_q(sl_2)$



Square (□) Lattice

$$\mathcal{H}_{\square} = i(q - q^{-1})(E + F)$$

Wiegmann & Zabrodin (1994)

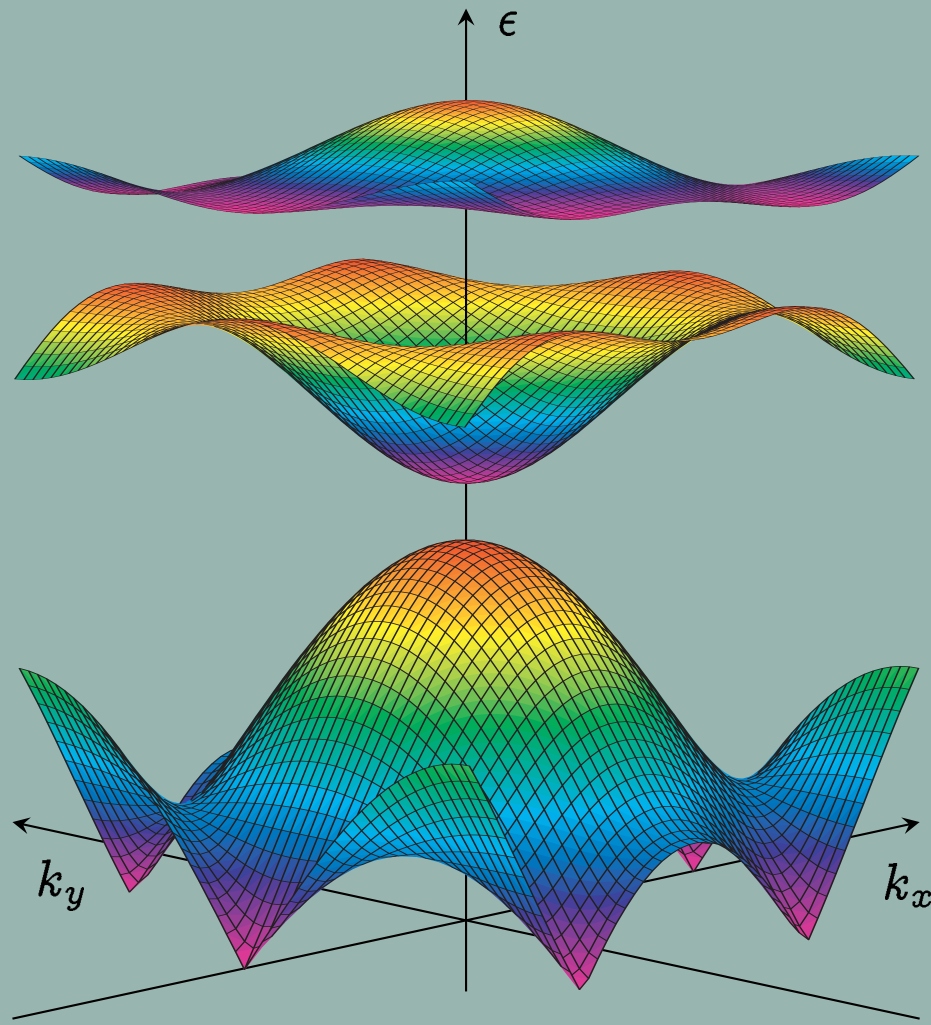


Honeycomb (⬡) Lattice ☛ SUSY QM

$$\mathcal{H}_{\hexagon} = \begin{pmatrix} 0 & \mathbb{I} + i(q - q^{-1})F \\ \mathbb{I} + i(q - q^{-1})E & 0 \end{pmatrix}$$

One-Particle Spectrum

$\Omega_N^\nu(\epsilon^2) = 0 \qquad N = 3$



General N

- Zero modes at special $\boldsymbol{k}$'s

$$\Omega_N^\nu(\epsilon^2) = \det(\Delta^\dagger \Delta - \mathbb{I} \cdot \epsilon^2)$$

$$\Omega_N^\nu(z) = \sum_{j=0}^{N} \omega_j(\boldsymbol{k}) z^j$$

Special $\boldsymbol{k}$'s ☛ $\omega_0(\boldsymbol{k}) = 0$

$$\Omega_N^\nu(z) = \omega_N z^N + \cdots + \omega_1 z$$

Reducibility of Characteristic Polynomials ($N = 9$)

$$\Omega_9^{\nu}(x) = (135 + 621w + 405w^2)x - (4239 + 1728w + 567w^2)x^2 +$$

$$+ (10314 + 1719w + 225w^2)x^3 - (10503 + 729w + 27w^2)x^4 +$$

$$+ (5643 + 135w)x^5 - (1719 + 9w)x^6 + 297x^7 - 27x^8 + x^9$$

$$w = 2\cos\left(\frac{2\pi\nu}{9}\right) \qquad w^3 - 3w + 1 = 0$$

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$$w = 2\cos\left(\frac{2\pi\nu}{9}\right) \qquad w^3 - 3w + 1 = 0$$

$$\Omega_9^\nu(x) = x \cdot (x - 4 + w^2) \cdot (x - 2 - w^2) \cdot P_3(x) \cdot Q_3(x)$$

$$P_3(x) = x^3 - 9x^2 + (14 + w + 2w^2)x - (3w + 3w^2)$$

$$Q_3(x) = x^3 - 12x^2 + (41 - 2w - w^2)x - (36 - 3w - 3w^2)$$

Coefficients of the factor polynomials carry the same structure as in $\Omega_9^\nu(x)$

Reducibility versus Unbroken SUSY

What may be the origin of the reducibility?

Reducibility occurs at special values of $\boldsymbol{k}$ where $\omega_0(\boldsymbol{k}) = 0$

$$\omega_0(\boldsymbol{k}) = 0 \quad \Rightarrow \quad \omega_N x^N + \cdots + \omega_1 x = 0 \quad \Rightarrow \quad \text{Zero mode}$$

Zero Mode $\longleftrightarrow$ Unbroken Supersymmetry

Unbroken Supersymmetry $\longleftrightarrow$ Characteristic Polynomials Reducible

Zero Modes

$$\begin{pmatrix} 0 & \Delta^\dagger \\ \Delta & 0 \end{pmatrix} \Psi = 0$$

$$\Psi_\blacktriangle = \begin{pmatrix} \xi_0 \\ 0 \end{pmatrix}, \qquad \Delta\,\xi_0 = 0$$

$$\Psi_\blacktriangledown = \begin{pmatrix} 0 \\ \zeta_0 \end{pmatrix}, \qquad \Delta^\dagger \zeta_0 = 0$$

$$\xi_0 = W \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_N \end{pmatrix}, \quad \zeta_0 = W \begin{pmatrix} Z_1^{-1} \\ Z_2^{-1} \\ \vdots \\ Z_N^{-1} \end{pmatrix}$$

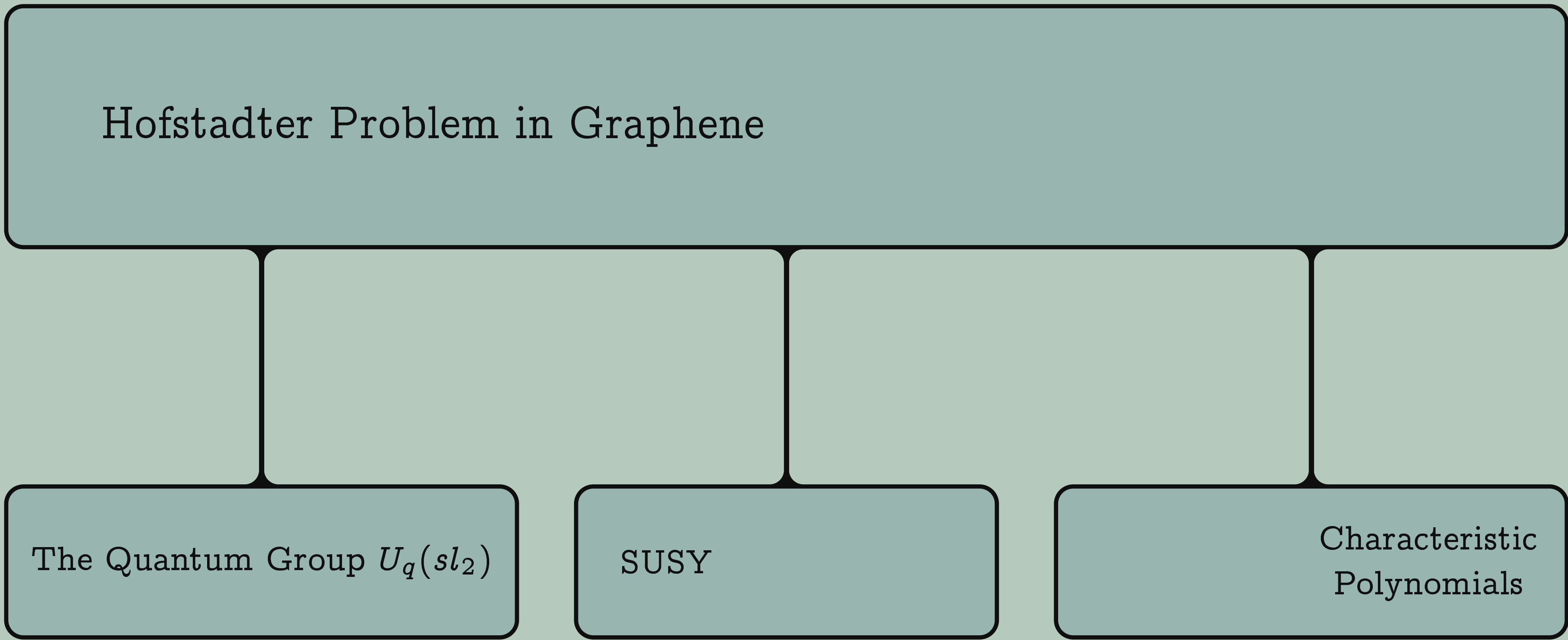
$$Z_j = t_1 t_2 \cdots t_j$$

$$t_n = 2\cos\left(\frac{\pi\nu}{N} n + \frac{\pi}{3N}\right)$$

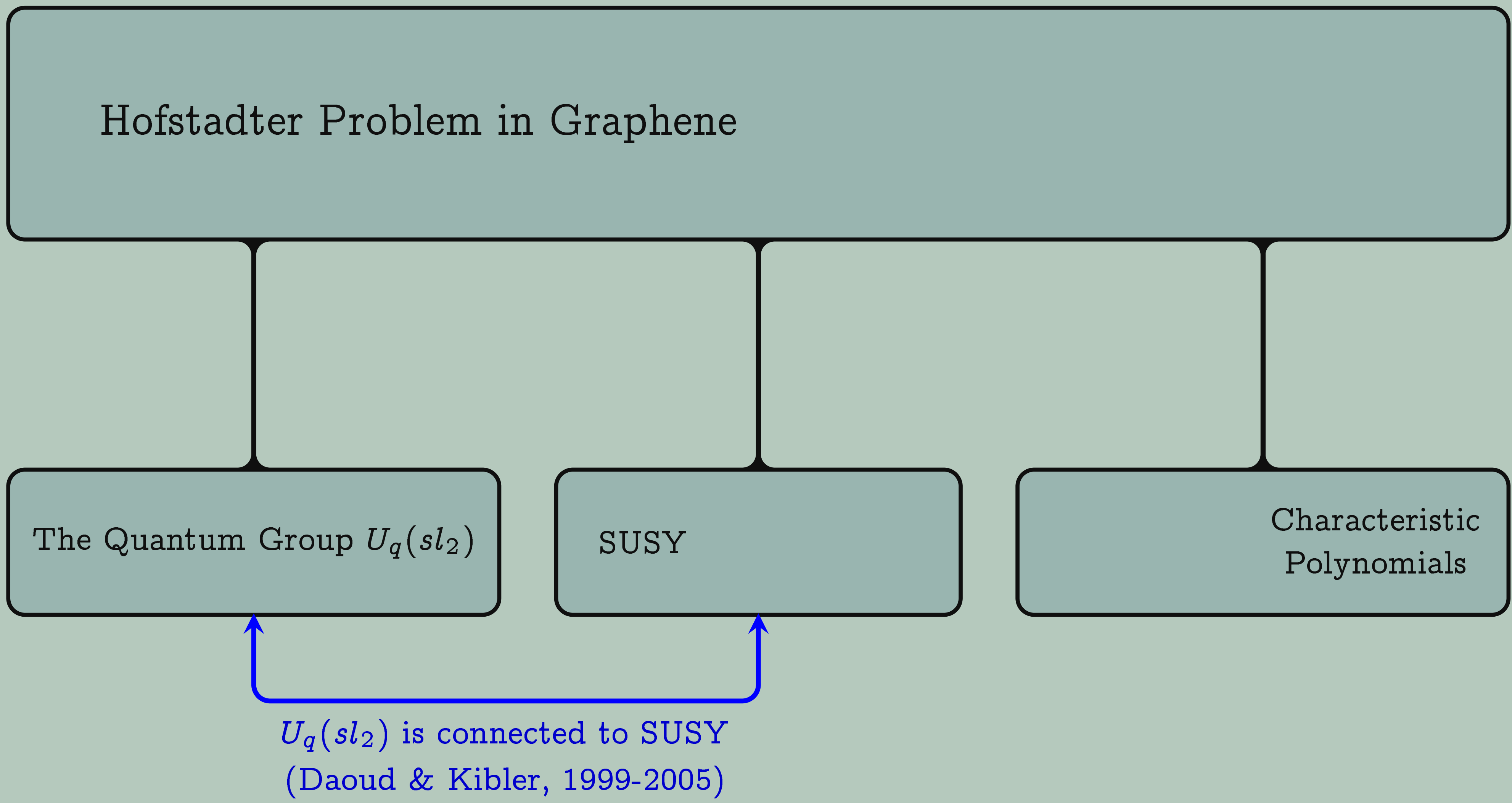
$$W_{n\kappa} = \frac{1}{\sqrt{N}} q^{n\kappa}$$

Zero mode sector is $U(1)$-degenerate: $\Psi_0 = e^{+i\alpha}\Psi_\blacktriangle + e^{-i\alpha}\Psi_\blacktriangledown$

Summary


Hofstadter Problem in Graphene
The Quantum Group $U_q(sl_2)$
SUSY
Characteristic Polynomials

Summary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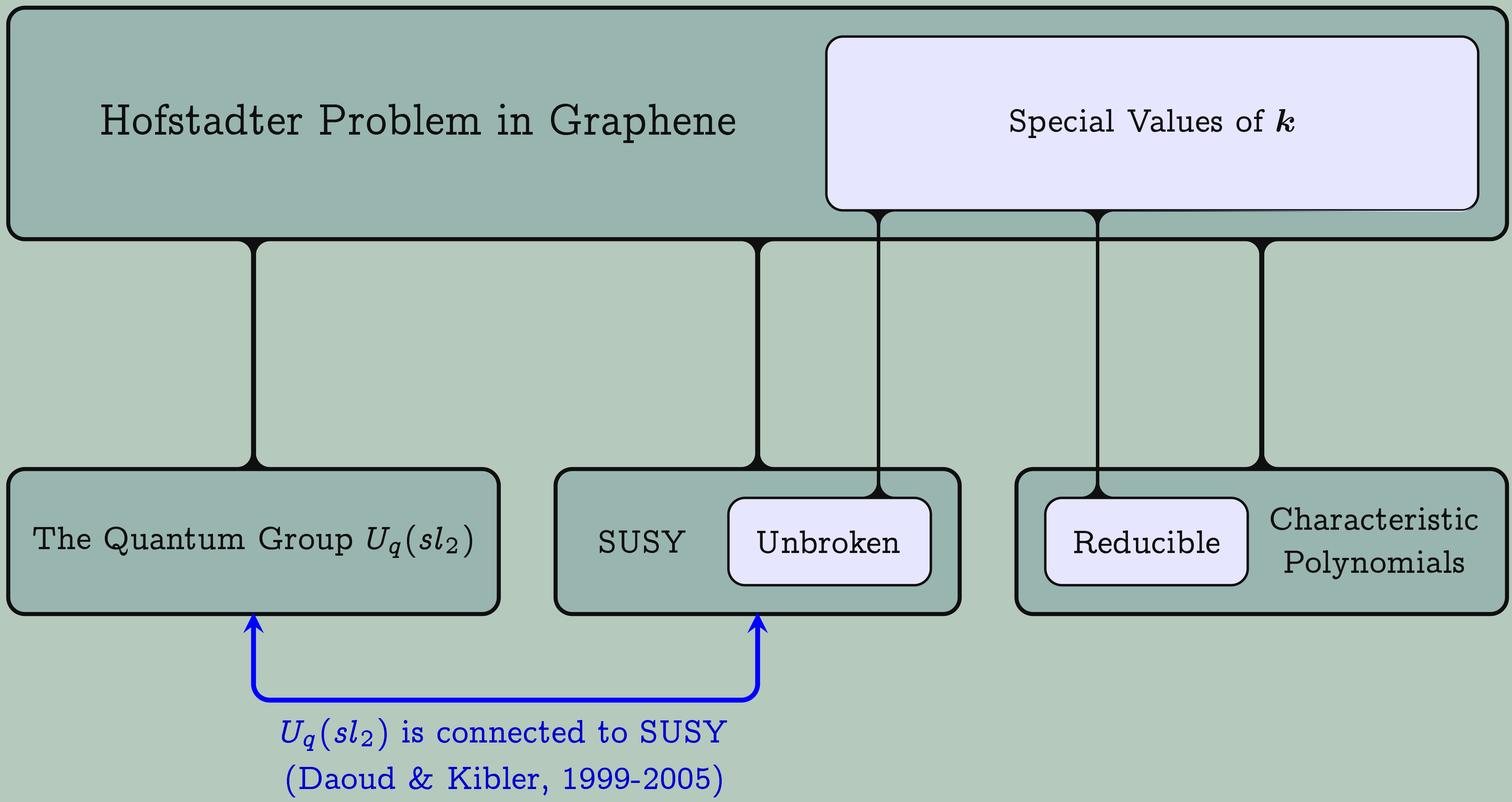
$U_q(sl_2)$ is connected to SUSY
(Daoud & Kibler, 1999-2005)

Summary



Hofstadter Problem in Graphene
Special Values of $\boldsymbol{k}$
The Quantum Group $U_q(sl_2)$
SUSY
Unbroken
Reducible
Characteristic Polynomials
$U_q(sl_2)$ is connected to SUSY
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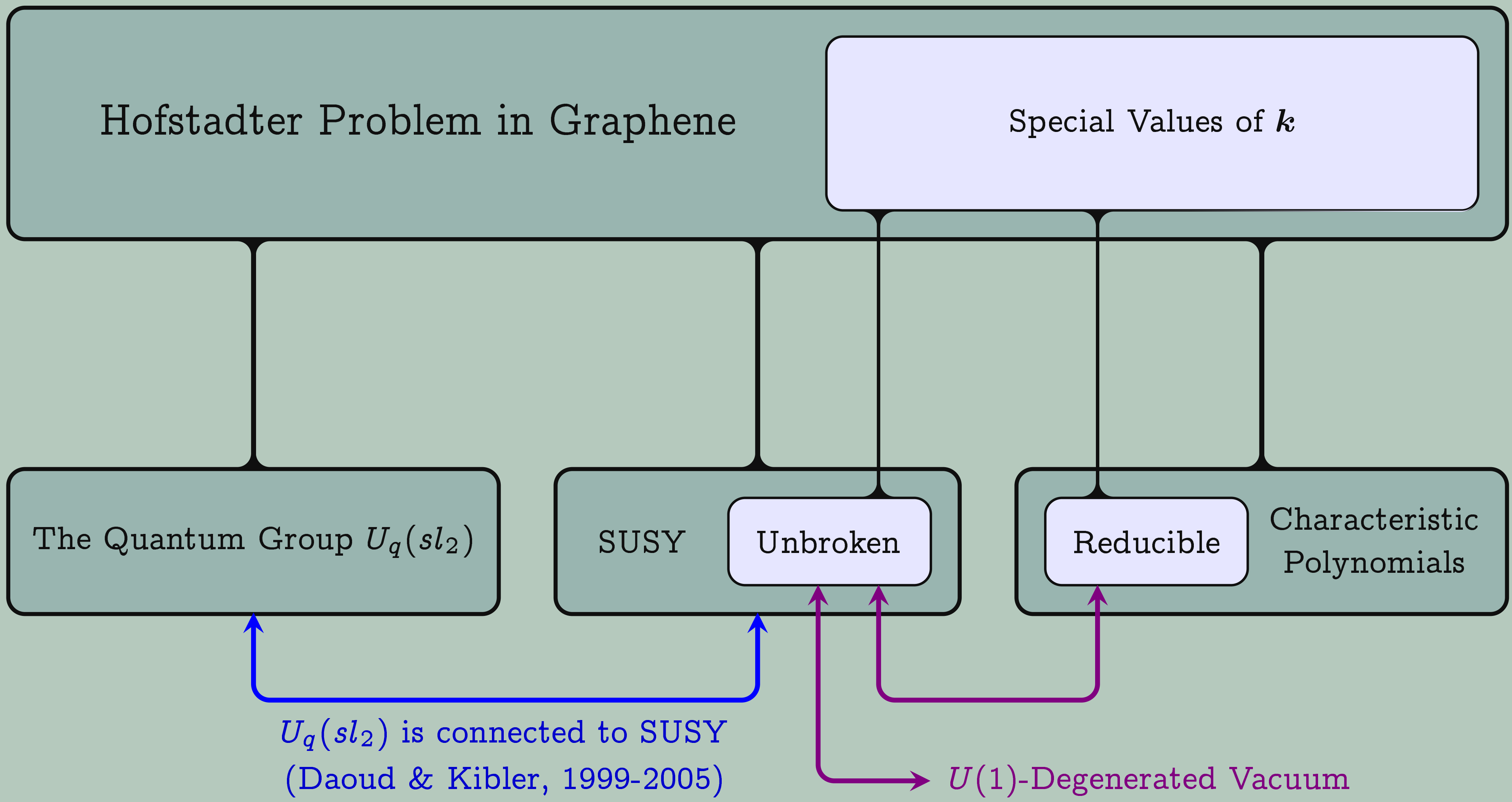
Summary



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The Quantum Group $U_q(sl_2)$
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$U_q(sl_2)$ is connected to SUSY
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$U(1)$-Degenerated Vacuum