



ELECTRON-IMPACT IONIZATION OF ATOMIC HYDROGEN: DYNAMICAL VARIATIONAL TREATMENT

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Outline

- Introduction.
- Formal theory of the variational principle.
- Application of variational principle for single and double ionization of atomic systems by electron impact.
- Effective, momentum-dependent, screened charges for the outgoing electrons.
- Physical characteristics of multiple ionization process
- Summary and conclusion

The main advantages of the variational method

- Purpose of this study is to show the ability of utilization of the variational principle as a simple and straightforward method of calculation for scattering processes.
- The novelty of the present treatment is that variation parameters are determined by using Hulthen-Kohn variational method for scattering amplitude.
- Such an approach enables to determine variational parameters mathematically in a rigorous way excluding any empirical assumptions .
- Calculation of the variational parameters is reduced to the finding of the approximate solution of the transcendental algebraic equation.

Why it is important to study electron impact ionization processes?

1. The study of the ionization processes (especially for a few electron atomic systems) represents the clue for understanding fundamental many particle interaction processes.
2. Many properties of plasma for example radiation property strongly depends on the ionization degree of the atoms and ions consist of plasma environment.
3. The proper understanding of ionization mechanisms and precise knowledge of corresponding cross sections are crucially important for keeping positive energy balance in fusion energy research .



Variational Principle for Discrete Spectrum Problems

For a Hamiltonian H and *any* normalizable function Ψ

$$L(\Psi) = \langle \Psi | H - \varepsilon | \Psi \rangle$$

$$\delta L = 0$$

Variational principle states that

- $\varepsilon \geq E_0$ E_0 the lowest energy eigenstate (ground state) of Hamiltonian
- $\varepsilon = E_0$ if and only if Ψ is exactly equal to the wavefunction of the ground state

Variational Principle for Continuum Spectrum Problems

$$L = \langle \Psi_f | H - E | \Psi_i \rangle$$

Ψ_i — initial state wave function

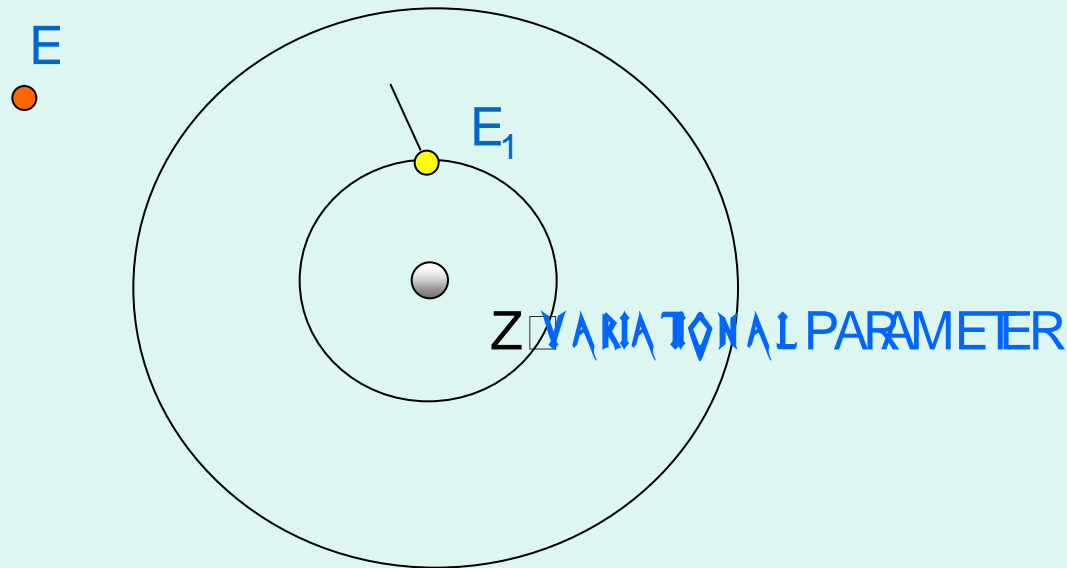
Ψ_f — final state wave function

$$\delta L = 0$$

Stationary Condition

$E > 0$ Continuum Spectrum

The ionization mechanism and effective the charge as a variational parameter



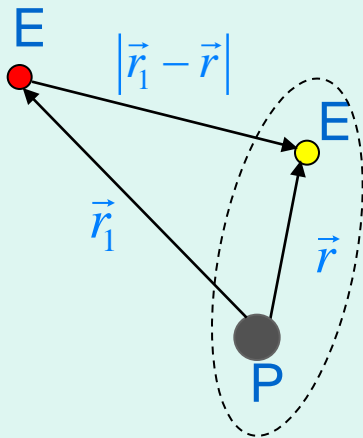
Variational Principle Applied for Ionization of Atomic Hydrogen

$$\hat{H} = -\frac{1}{2}\Delta_1 + \hat{H}_0(\vec{r}) + V,$$

Total hamiltonian.

$$V = -1/r_1 + 1/|\vec{r}_1 - \vec{r}|$$

Interaction Potential bewteen the incident electron and target.



FUNCTIONAL

$$\tilde{L}(\vec{n}_i, -\vec{n}_s) = \iint \Psi_f(-\vec{n}_s, \vec{r}_1, \vec{r})(\hat{H} - E)\Psi_i(\vec{n}_i, \vec{r}_1, \vec{r})d\vec{r}_1d\vec{r}$$

$\Psi_i(\vec{n}_i, \vec{r}_1, \vec{r})$ — INITIAL WAVE FUNCTION

$\Psi_f(-\vec{n}_s, \vec{r}_1, \vec{r})$ — FINAL WAVE FUNCTION

E — TOTAL ENERGY OF THE SYSTEM

Proper Asymptotic Form Corresponding to the Process Under Consideration

$$\Psi_i \approx \frac{e^{i\vec{k}_i \vec{r}_1}}{(2\pi)^{3/2}} \varphi_i(\vec{r}) + \sum_j F_{ij}(\vec{n}_i, \vec{n}) \varphi_j(\vec{r}) \frac{e^{ik_j r_1}}{r_1} + \int F_{i\lambda}(\vec{n}_i, \vec{n}) \Phi_\lambda(\vec{r}) \frac{e^{ik_\lambda r}}{r} d\lambda,$$

$$\Psi_f \approx \frac{e^{-i\vec{k}_s \vec{r}_1}}{(2\pi)^{3/2}} \varphi_f(\vec{r}) + \sum_j F_{fj}(-\vec{n}_s, \vec{n}) \varphi_j^*(\vec{r}) \frac{e^{ik_j r_1}}{r_1} + \int F_{f\lambda}(-\vec{n}_s, \vec{n}) \Phi_\lambda^*(\vec{r}) \frac{e^{ik_\lambda r_1}}{r_1} d\lambda.$$

$F_{ij}(\vec{n}_i, \vec{n})$ AND $F_{i\lambda}(\vec{n}_i, \vec{n})$ ARE AMPLITUDES OF THE OUTGOING SPHERICAL WAVES PROPAGATING IN \vec{n} DIRECTION

$$\int \varphi_i^*(\vec{r}) \square H_0 \varphi_i(\vec{r}) d\vec{r} + \frac{k_i^2}{2} = E$$

THIS CONDITION PROVIDES THE CONVERGENCE OF THE FUNCTIONAL

\vec{k}_i — THE WAVE VECTOR OF THE INCIDENT ELECTRON

Variation of Functional

$$\delta\tilde{L}(\vec{n}_i, -\vec{n}_s) = -\frac{1}{2} \iint \left[\Psi_f \Delta(\delta\Psi_i) - \delta\Psi_i \Delta(\Psi_f) \right] d\vec{r}_1 d\vec{r}$$

TRANSFORMATION INTO A SURFACE INTEGRAL BY GREEN'S THEOREM

$$\delta\tilde{L}(\vec{n}_i, -\vec{n}_s) = -\frac{1}{2} \int_S \left[\Psi_f \frac{\partial}{\partial r_1}(\delta\Psi_i) - \delta\Psi_i \frac{\partial}{\partial r_1}(\Psi_f) \right] d\vec{r} dS$$

THE CONDITION AFTER SUBSTITUTING ASYMPTOTIC WAVE FUNCTIONS AND PERFORMING THE INTEGRATION

$$\delta\tilde{L}(\vec{n}_i, -\vec{n}_s) = (2\pi)^{-1/2} \left[F_{if}(\vec{n}_i, \vec{n}_s) - F_{fi}(-\vec{n}_s, -\vec{n}_i) + \delta F_{if}(\vec{n}_i, \vec{n}_s) \right]$$

ACCORDING TO RECIPROCITY PROPERTY OF AMPLITUDE

$$F_{if}(\vec{n}_i, \vec{n}_s) = F_{fi}(-\vec{n}_s, -\vec{n}_i)$$

Stationary Property of the Amplitude

$$\delta \tilde{L}(\vec{n}_i, -\vec{n}_s) = (2\pi)^{-1/2} \delta F_{if}(\vec{n}_i, \vec{n}_s)$$

OR IN EQUIVALENTLY

$$\delta L(\vec{n}_i, -\vec{n}_s) = 0,$$

$$L(\vec{n}_i, -\vec{n}_s) = F_{if}(\vec{n}_i, \vec{n}_s) - \sqrt{2\pi} \tilde{L}(\vec{n}_i, -\vec{n}_s)$$

FUNCTIONAL IS STATIONARY WITH RESPECT TO SMALL DEVIATIONS OF THE WAVEFUNCTIONS

$$L(\vec{n}_i, -\vec{n}_s) \approx L^{(t)}(\vec{n}_i, -\vec{n}_s) = F_{if}^{(t)}(\vec{n}_i, \vec{n}_s) - \sqrt{2\pi} \tilde{L}^{(t)}(\vec{n}_i, -\vec{n}_s)$$

AMPLITUDE IS DETERMINED TO BE EQUAL TO FUNCTIONAL

$$F_{if}(\vec{n}_i, \vec{n}_s) = F_{if}^{(t)}(\vec{n}_i, \vec{n}_s) - \sqrt{2\pi} \iint \Psi_f^{(t)}(-\vec{n}_s, \vec{r}_1, \vec{r}) \left(\hat{H} - E \right) \Psi_i^{(t)}(\vec{n}_i, \vec{r}_1, \vec{r}) d^3r_1 d^3r$$

$\Psi_i^{(t)}$ — INITIAL TRIAL WAVEFUNCTION

$\Psi_f^{(t)}$ — FINAL TRIAL WAVEFUNCTION

Choice of the Trial Functions

$$\Psi_i^{(t)} = \frac{e^{i\vec{k}_i \vec{r}_1}}{(2\pi)^{3/2}} \varphi_0(\vec{r}) \quad \varphi_0(\vec{r}) = e^{-r} / \sqrt{\pi} \quad \text{GROUND STATE OF HYDROGEN}$$

$$\Psi_f^{(t)} = f_{-\vec{k}_i}(z, \vec{r}_1) f_{\vec{k}}(\vec{r}) \quad \text{Z IS AN EFFECTIVE CHARGE SEEN BY THE SCATTERED ELECTRON}$$

$$f_{\vec{k}}(\vec{r}) = \frac{e^{\pi/2k}}{(2\pi)^{3/2}} \Gamma\left(1 + \frac{i}{k}\right) e^{i\vec{k}\vec{r}} F\left(-\frac{i}{k}, 1, -i(kr + \vec{k}\vec{r})\right)$$

$$f_{-\vec{k}_s}(z, \vec{r}_1) = \frac{e^{z\pi/2k_s}}{(2\pi)^{3/2}} \Gamma\left(1 - \frac{iz}{k_s}\right) e^{-i\vec{k}_s \vec{r}_1} F\left(\frac{iz}{k_s}, 1, i(k_s r_1 + \vec{k}_s \vec{r}_1)\right)$$

COMPLEX CONTINUUM FUNCTIONS

$F(a, b, x)$ CONFLUENT HYPERGEOMETRIC FUNCTION

$\Gamma(x)$ EULER'S GAMMA FUNCTION

Ionization Amplitude

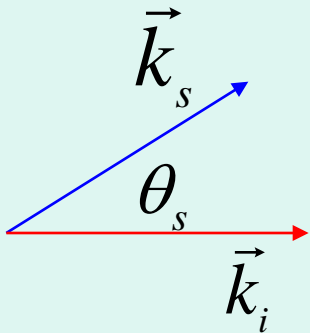
$$F_{if}(\vec{n}_i, \vec{n}_s) = I_{el}(z) - I(z)$$

$$I_{el}(z) = \frac{z}{2k_s^2 \sin^2(\theta_s/2)} \frac{\Gamma(1 - iz/k_s)}{\Gamma(1 + iz/k_s)} e^{i\frac{2z}{k_s} \ln \sin(\theta_s/2)}$$

ELASTIC SCATTERING

$$I(z) = \frac{1}{2\pi} \int f_{\vec{k}}^*(\vec{r}) \int f_{-\vec{k}_s}(z, \vec{r}_1) \frac{1}{|\vec{r}_1 - \vec{r}|} e^{i\vec{k}_i \cdot \vec{r}_1} d\vec{r}_1 \varphi_0(\vec{r}) d\vec{r}$$

IONIZATION AMPLITUDE



θ_s — SCATTERING ANGLE

Effective Charge Seen by the Scattered Electron

PROVIDED THAT $F_{if}(\vec{n}_i, \vec{n}_s) \neq 0$

EQUATION FOR EFFECTIVE CHARGE χ

$$d \left| F_{if}(\vec{n}_i, \vec{n}_s) \right|^2 / dz = 0$$

TRANSCENDENTAL ALGEBRAIC EQUATION

$$\left(1 - 2k_s^2 \sin^2 \frac{\theta_s}{2} \frac{d\chi}{dz} \right) \chi - 2k_s^2 \sin^2 \frac{\theta_s}{2} \chi + 2k_s^4 \sin^4 \frac{\theta_s}{2} \frac{d|I(z)|^2}{dz} = 0,$$

$$\chi(z) = \text{Re} \left[\frac{\Gamma(1 + iz/k_s)}{\Gamma(1 - iz/k_s)} e^{-i \frac{2z}{k_s} \ln \sin(\theta_s/2)} I(z) \right]$$

Solution for small values of z

$$I(z) = \frac{e^{z\pi/2k_s}}{2^{1/2} \pi^{3/2}} \Gamma(1 - iz/k_s) \left[q^{-2} A(\vec{q}) + zB(z, \vec{q}) \right]$$

$$A(\vec{q}) \text{ — } \text{BORN AMPLITUDE}$$

$$B(z, \vec{q}) = \frac{i}{(2\pi)^3 k_s} \int A(\vec{p}) \int e^{i(\vec{q}-\vec{p})\vec{r}_1} \Phi(\alpha, x) d\vec{r}_1 \frac{d\vec{p}}{p^2}$$

$$\Phi(\alpha, x) = \frac{x}{1!} + \frac{\alpha}{1 \cdot 2} \frac{x^2}{2!} + \frac{\alpha(\alpha+1)}{1 \cdot 2 \cdot 3} \frac{x^3}{3!} + \dots$$

$$\vec{q} = \vec{k}_i - \vec{k}_s \text{ MOMENTUM TRANSFER} \quad \alpha = 1 + iz/k_s \quad x = i(k_s r_1 + \vec{k}_s \vec{r}_1)$$

Solution for small values of z

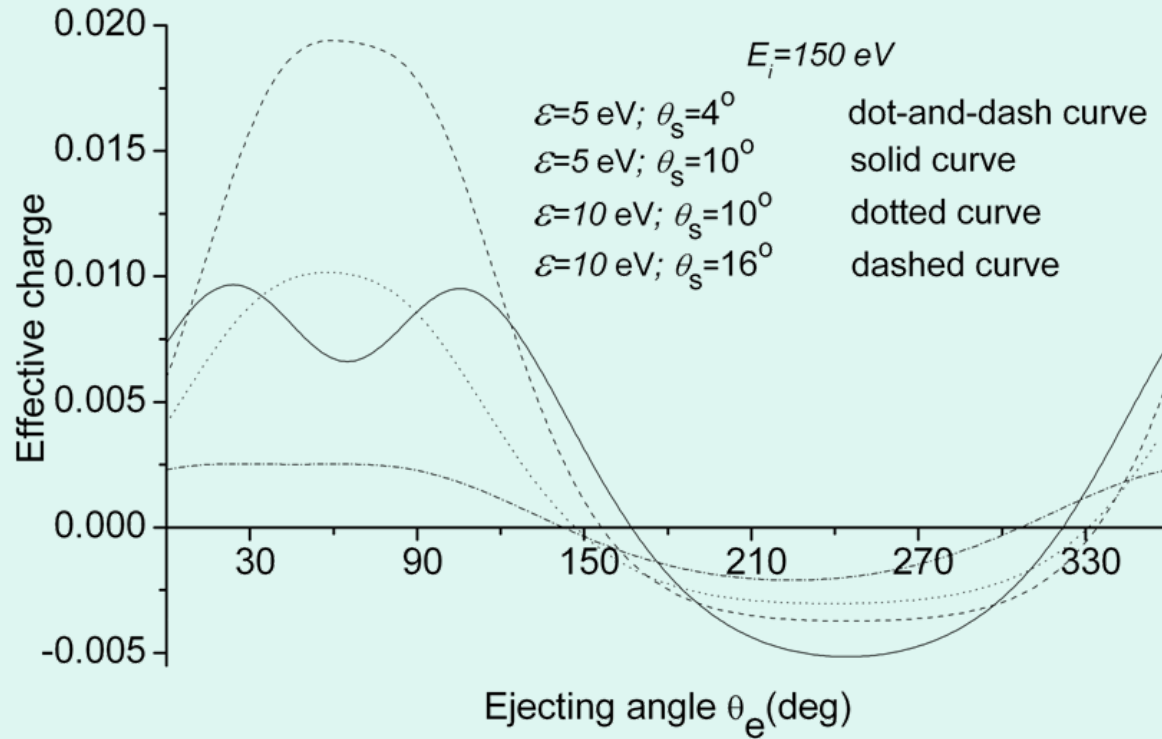
$$z = g \frac{q^{-2} \operatorname{Re} A - gG_1}{1 - gG_2}$$

$$g = 2^{1/2} \pi^{-3/2} k_s^2 \sin^2(\theta_s / 2)$$

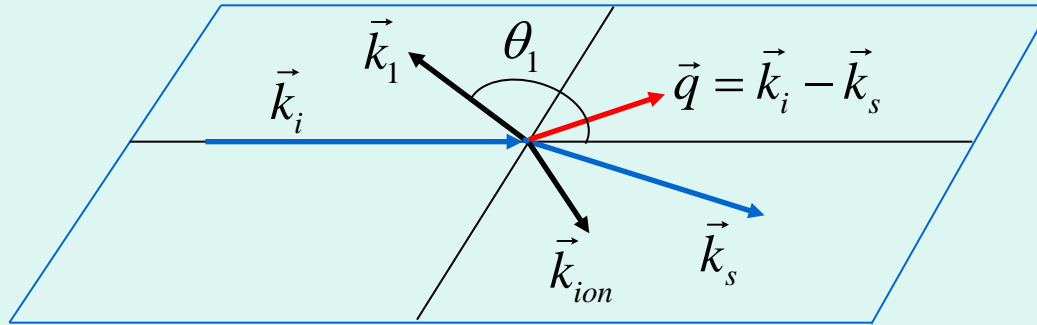
$$G_1 = 2^{-1} k_s^{-1} q^{-2} \left[\pi q^{-2} |A|^2 + 2k_s \operatorname{Re}(AB^*) \right]$$

$$G_2 = k_s^{-1} \left[\pi q^{-2} \operatorname{Re} A + 2q^{-2} (C + 2 \ln \sin(\theta_s / 2)) \operatorname{Im} A + 2k_s \operatorname{Re} B \right].$$

Behavior of z Versus Scattering Angle



Coplanar Scattering Geometry



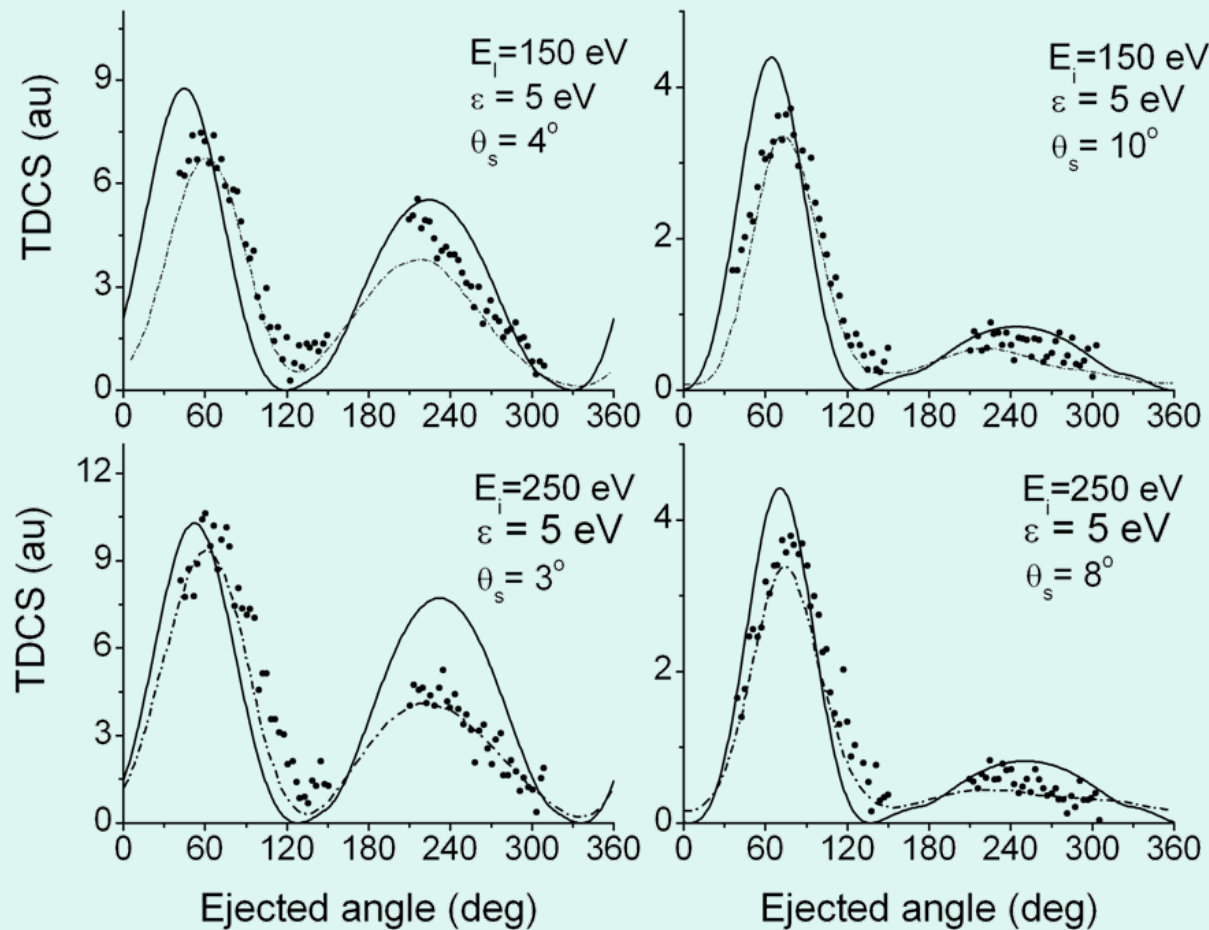
$$E_i = E_s + E_1 + E_{bind}$$

Energy conservation

$$\vec{k}_i = \vec{k}_s + \vec{k}_1 + \vec{k}_{ion}$$

Momentum conservation

Differential Cross Section for Ionization of Atomic Hydrogen



P Defrance, T Kereselidze, J Lecointre, Z S Machavariani Eur.Phys. J. D (2011) (Accepted for publication)

Summary and Conclusion

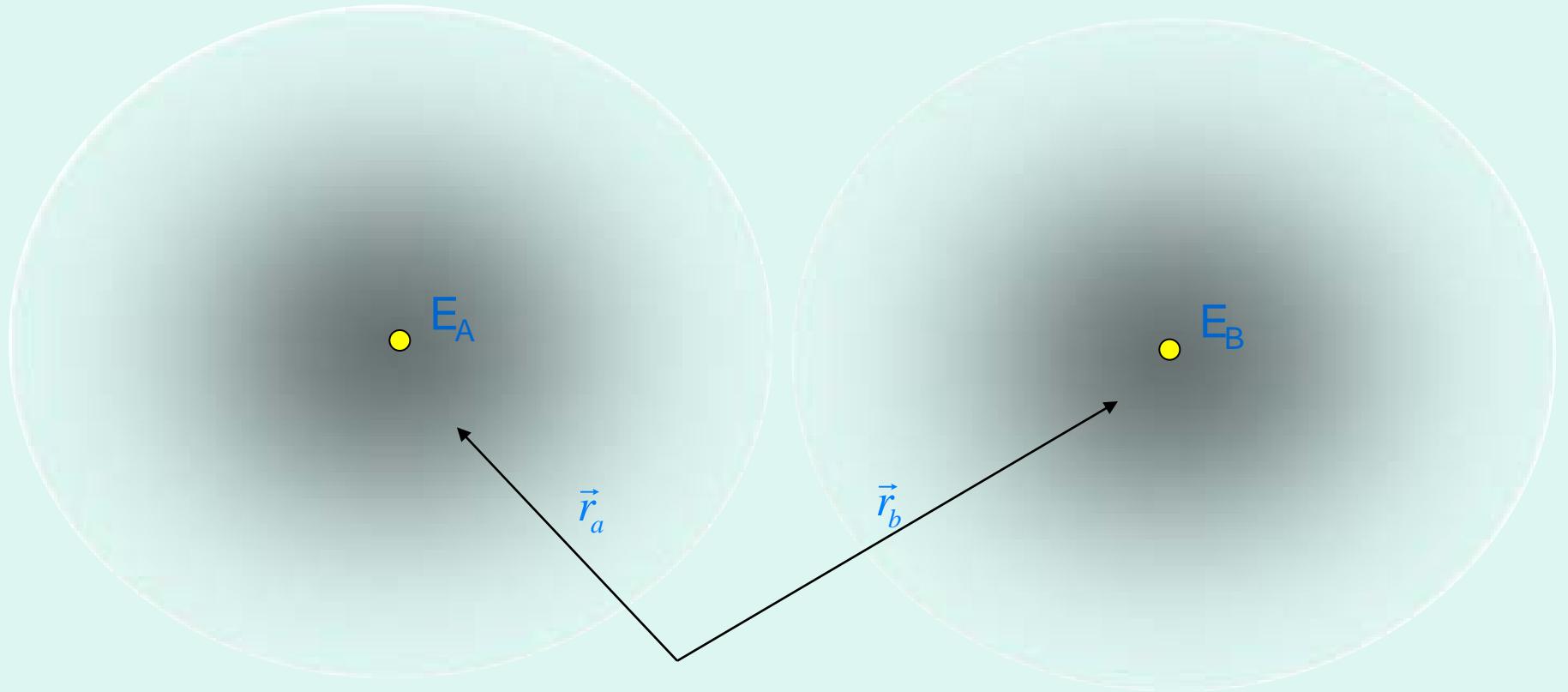
- The examination of obtained results shows that elaborated approach gives for the effective charge values which are reasonable and have clear physical meaning .
- Effective charges are introduced naturally as the variational parameters.
- Determination of the effective charges is based on the exact quantum-mechanical principles, without any empirical assumptions.
- The equation for the effective charge can be solved analytically.
- Variational principle can be applied to study electron-impact double, triple and ect. Ionization processes.

List of major Scientific Publications

- 1) P Defrance, T Kereselidze, J Lecointre, Z S Machavariani *Eur.Phys. J. D* (2011) DOI: 10.1140/epjd/e2011-20285-x
- 2) P Defrance, J J Jureta, T Kereselidze, J Lecointre, Z S Machavariani *J. Phys. B: At. Mol. Opt. Phys.* **42** 025202 (2009)
- 3) J. S. Alexander, A. C. Laforge, A. Hasan, Z. S. Machavariani, M. F. Ciappina, R. D. Rivarola, D. H. Madison, and M. Schulz *Phys. Rev. A* **78**, 060701(R) (2008)
- 4) E Bahati, H Cherkani-Hassani, P Defrance, J J Jureta, T Kereselidze, Z Machavariani and I L Noselidze *J. Phys. B: At. Mol. Opt. Phys.* **38** 1261-1277 (2005)
- 5) P Defrance, T M Kereselidze, Z S Machavariani *Nuclear Instruments and Methods in Physics Research B (NIM B)* v205 pg405-409 (2003) (North Holland)
- 6) P Defrance, T Kereselidze, I L Noselidze, M F Tsulukidze *J. Phys. B: At. Mol. Opt. Phys.* **35** 1799 (2002)
- 7) P Defrance, T Kereselidze, I L Noselidze, M F Tsulukidze *J. Phys.B. At. Mol. Opt. Phys.* **34** 4957 (2001)
- 8) P Defrance, T M Kereselidze, Z S Machavariani, I L Noselidze *J. Phys.B. At. Mol. Opt. Phys.* **33** 4323 (2000)

Thank you for your attention!

The electron-electron correlation effect



$$\rho(\vec{r}_a, \vec{r}_b) \neq \rho(\vec{r}_a)\rho(\vec{r}_b)$$

$$\rho(\vec{r}_a, \vec{r}_b) d\vec{r}_a d\vec{r}_b$$

EXPLICIT EXPRESSIONS FOR THE EFFECTIVE CHARGES SUGGESTED BY GIZ et al

$$Z_1^* = Z - (1 - \tilde{Z}_{12}) \frac{k_1^2}{2k_{12}(k_1 + k_2)},$$

$$Z_2^* = Z - (1 - \tilde{Z}_{12}) \frac{k_2^2}{2k_{12}(k_1 + k_2)},$$

$$k_{12} = \sqrt{k_1^2 - 2k_1k_2 \cos(\vec{k}_1 \vec{k}_2) + k_2^2}, \quad \tilde{Z}_{12} = 1 - \frac{4k_{12}^2}{(k_1 + k_2)^2}.$$

Comment DERIVATION OF GIZ AND CO WORKERS IS OBTAINED IN THE SOMEWHAT
INTUITIVE MANNER AND ACCORDINGLY IS NOT MATHEMATICALLY STRICT

Reference:

GIZ, R. GIZ, GIZ, M. HAIDER AND J. S. BRIGGS *J. Phys. B: At. Mol. Opt. Phys.* **38** 1569

VARIATIONAL PRINCIPLE FOR DOUBLE IONIZATION OF HELIUM-LIKE SYSTEMS BY ELECTRONS

$$\Psi_i \approx \frac{e^{i\vec{k}_i \vec{r}}}{(2\pi)^{3/2}} \Phi_i(\vec{r}_1, \vec{r}_2) + \sum_j F_{ij}(\vec{n}_i, \vec{n}) \Phi_j(\vec{r}_1, \vec{r}_2) \frac{e^{ik_j r}}{r} + \int F_{i\varepsilon}(\vec{n}_i, \vec{n}) \Phi_\varepsilon(\vec{r}_1, \vec{r}_2) \frac{e^{ik_\varepsilon r}}{r} d\varepsilon,$$

$$\Psi_f \approx \frac{e^{-i\vec{k}_s \vec{r}}}{(2\pi)^{3/2}} \Phi_f(\vec{r}_1, \vec{r}_2) + \sum_j F_{fj}(-\vec{n}_s, \vec{n}) \Phi_j^*(\vec{r}_1, \vec{r}_2) \frac{e^{ik_j r}}{r} + \int F_{f\varepsilon}(-\vec{n}_s, \vec{n}) \Phi_\varepsilon^*(\vec{r}_1, \vec{r}_2) \frac{e^{ik_\varepsilon r}}{r} d\varepsilon.$$

$$\tilde{L}(\vec{n}_i, -\vec{n}_s) = \int \Psi_f \left(\hat{H} - E \right) \Psi_i d\vec{r} d\vec{r}_1 d\vec{r}_2$$

$$\delta \tilde{L}(\vec{n}_i, -\vec{n}_s) = -\frac{1}{2} \int_S \int \int \left[\Psi_f \frac{\partial}{\partial r} (\delta \Psi_i) - \delta \Psi_i \frac{\partial}{\partial r} (\Psi_f) \right] d\vec{r}_1 d\vec{r}_2 dS$$

$$\delta L(\vec{n}_i, -\vec{n}_s) = \delta \left[F_{if}(\vec{n}_i, \vec{n}_s) - \sqrt{2\pi} \tilde{L}(\vec{n}_i, -\vec{n}_s) \right] = 0$$

$$F_{if}(\vec{n}_i, \vec{n}_s) = L(\vec{n}_i, -\vec{n}_s) \approx F_{if}^{(t)}(\vec{n}_i, \vec{n}_s) - \sqrt{2\pi} \tilde{L}^{(t)}(\vec{n}_i, -\vec{n}_s)$$

$$\tilde{L}^{(t)}(\vec{n}_i, -\vec{n}_s) = \int \Psi_f^{(t)} \left(\hat{H} - E \right) \Psi_i^{(t)} d\vec{r} d\vec{r}_1 d\vec{r}_2$$

$\Psi_f^{(t)}$ and $\Psi_i^{(t)}$ are the trial wavefunctions

$$\frac{\partial L(\alpha_1, \alpha_2, \dots, \alpha_n)}{\partial \alpha_j} = 0$$

$$(j = 1, 2, \dots, n)$$

Reference:

M. F. MOTT AND H. S. M. MASSEY, THE THEORY OF ATOMIC COLLISIONS, OXFORD UNIVERSITY PRESS, 1965

CHOICE OF THE TRIAL WAVE FUNCTIONS FOR DOUBLE IONIZATION OF HELIUM

$$\Psi_i^{(t)} = \frac{e^{i\vec{k}_i \vec{r}}}{(2\pi)^{3/2}} \Phi_i^{(t)}(\vec{r}_1, \vec{r}_2) \quad \text{--- INITIAL TRIAL WAVE FUNCTION}$$

$$\Psi_f^{(t)} = \frac{e^{i\vec{k}_f \vec{r}}}{(2\pi)^{3/2}} \Phi_f^{(t)}(\vec{r}_1, \vec{r}_2) \quad \text{--- FINAL TRIAL WAVE FUNCTION}$$

$$\Phi_i^{(t)}(\vec{r}_1, \vec{r}_2) = C_i e^{-\alpha(r_1+r_2)} \quad \alpha = 1.6875$$

$$\Phi_f^{(t)}(\vec{r}_1, \vec{r}_2) = \varphi_f(\vec{r}_1, \vec{r}_2) - S^* \Phi_i(\vec{r}_1, \vec{r}_2) \quad S = \langle \varphi_f | \Phi_i \rangle$$

$$\varphi_f(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} \left(f_{\vec{k}_1}(\vec{r}_1) f_{\vec{k}_2}(\vec{r}_2) + f_{\vec{k}_1}(\vec{r}_2) f_{\vec{k}_2}(\vec{r}_1) \right)$$

$$f_{\vec{k}_j}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{\pi Z_j^*/2k_j} \Gamma\left(1 + \frac{iZ_j^*}{k_j}\right) e^{i\vec{k}_j \vec{r}} F\left(-\frac{iZ_j^*}{k_j}, 1, -i(k_j r + \vec{k}_j \vec{r})\right)$$

▷ DOUBLE IONIZATION AMPLITUDE

$$F_{if}(\vec{n}_i, \vec{n}_s) = -\frac{2C_i}{\pi^{3/2} q^2} \left[I(q, Z_1^*, \vec{k}_1) J(Z_2^*, k_2) + I(q, Z_2^*, \vec{k}_2) J(Z_1^*, k_1) \right]$$

$$I(q, Z_j^*, \vec{k}_j) = \int f_{\vec{k}_j}^*(\vec{r}) \left(e^{i\vec{q}\vec{r}} - \frac{G(q)}{2} \right) e^{-\alpha r} d\vec{r} \quad \text{--- DIRECTIONIZATION}$$

$$J(Z_j^*, k_j) = \int f_{\vec{k}_j}^*(\vec{r}) e^{-\alpha r} d\vec{r} \quad \text{--- IONIZATION DUE TO SUDDEN CHANGE IN POTENTIAL}$$

$$G(q) = \iint \Phi_i^* \left(e^{i\vec{q}\vec{r}_1} + e^{i\vec{q}\vec{r}_2} \right) \Phi_i d\vec{r}_1 d\vec{r}_2$$

Reference:

- 1 ▷ DEFRANCE □ T KERESEDDZE □ Z M ACHAVARIANI J MEBONIA 1999
J. Phys. B: At. Mol. Opt. Phys. **32** 2227
- 2 ▷ DEFRANCE □ T KERESEDDZE □ Z M ACHAVARIANI AND
INQSEDDZE 2000 *J.Phys.B: At. Mol. Opt. Phys.* **33** 4323

▷ DOUBLE IONIZATION AMPLITUDE AT SMALL MOMENTUM TRANSFER

$$F_{if}(\vec{n}_i, \vec{n}_s) = -\frac{2C_i}{\pi^{3/2} q^2} h(Z_1^*, k_1) h(Z_2^*, k_2) \left\{ \left[\operatorname{Re} M^{(1)}(Z_1^*, Z_2^*, \vec{k}_1, \vec{k}_2) + i \operatorname{Im} M^{(1)}(Z_1^*, Z_2^*, \vec{k}_1, \vec{k}_2) \right] q + \left[\operatorname{Re} M^{(2)}(Z_1^*, Z_2^*, \vec{k}_1, \vec{k}_2) + i \operatorname{Im} M^{(2)}(Z_1^*, Z_2^*, \vec{k}_1, \vec{k}_2) \right] q^2 + \dots \right\}$$

$$\operatorname{Re} M^{(1)} = 2 \left[k_1 (Z_1^* - 2\alpha)(Z_2^* - \alpha) \frac{\cos \theta_1}{\alpha^2 + k_1^2} + k_2 (Z_2^* - 2\alpha)(Z_1^* - \alpha) \frac{\cos \theta_2}{\alpha^2 + k_2^2} \right]$$

$$\operatorname{Im} M^{(1)} = -2 \left[Z_1^* (Z_1^* - 2\alpha)(Z_2^* - \alpha) \frac{\cos \theta_1}{\alpha^2 + k_1^2} + Z_2^* (Z_2^* - 2\alpha)(Z_1^* - \alpha) \frac{\cos \theta_2}{\alpha^2 + k_2^2} \right]$$

$$\cos \vartheta_1 = \cos(\vec{q}\vec{k}_1) \quad \cos \vartheta_2 = \cos(\vec{q}\vec{k}_2) \quad h(Z_j^*, k_j) = J^{(0)}(Z_j^*, k_j) N(Z_j^*, k_j)$$

$$J^{(0)}(Z_j^*, k_j) = -\frac{8\pi}{(\alpha^2 + k_j^2)^2} \exp\left(-\frac{Z_j^*}{k_j} \operatorname{arctg} \frac{2\alpha k_j}{\alpha^2 - k_j^2}\right) \quad N(Z_j^*, k_j) = \frac{e^{\pi Z_j^*/2k_j}}{(2\pi)^{3/2}} \Gamma(1 - iZ_j^*/k_j)$$

Reference:

□ □ ▷ EFRANÇE □ T KERESELDZE □ N QSELDZE AND M
TZULKIDZE 2001 *J.Phys.B: At. Mol. Opt. Phys.* **34** 4957

EQUATIONS FOR THE EFFECTIVE CHARGES IN THE QUADRUPOLE APPROXIMATION

$$\begin{aligned} \text{Im } M^{(1)} \left[\frac{\partial \text{Im } M^{(1)}}{\partial Z_1^*} + a(Z_1^*, k_1) \text{Im } M^{(1)} \right] + \text{Re } M^{(1)} \left[\frac{\partial \text{Re } M^{(1)}}{\partial Z_1^*} + a(Z_1^*, k_1) \text{Re } M^{(1)} \right] + q Q = 0 \\ \text{Im } M^{(1)} \left[\frac{\partial \text{Im } M^{(1)}}{\partial Z_2^*} + a(Z_2^*, k_2) \text{Im } M^{(1)} \right] + \text{Re } M^{(1)} \left[\frac{\partial \text{Re } M^{(1)}}{\partial Z_2^*} + a(Z_2^*, k_2) \text{Re } M^{(1)} \right] + q Q = 0 \end{aligned}$$

$$\begin{aligned} Q_j = \text{Im } M^{(1)} \frac{\partial \text{Im } M^{(2)}}{\partial Z_j^*} + \text{Im } M^{(2)} \frac{\partial \text{Im } M^{(1)}}{\partial Z_j^*} + \text{Re } M^{(1)} \frac{\partial \text{Re } M^{(2)}}{\partial Z_j^*} + \\ \text{Re } M^{(2)} \frac{\partial \text{Re } M^{(1)}}{\partial Z_j^*} + 2a(Z_j^*, k_j) \left[\text{Im } M^{(1)} \text{Im } M^{(2)} + \text{Re } M^{(1)} \text{Re } M^{(2)} \right] \end{aligned}$$

$$a(Z_j^*, k_j) = -\frac{1}{k_j} \text{arctg} \left(\frac{2\alpha k_j}{\alpha^2 - k_j^2} \right)$$

Fig. 2 THE TOTAL CROSS SECTION FOR DOUBLE IONIZATION OF HELIUM BY FAST ELECTRONS GIVEN IN THE BETHE PLOT FORM [BLACK LINE [PRESENT RESULT [RED LINE [PREVIOUS RESULT [12 [EXPERIMENTAL DATA [