(I) Analytic and Model-Independent Determination of the Magnetic moments of the Δ Resonances & Deuteron in the πN & NN Bremsstrahlung $\pi N \to \gamma' \pi' N'$ & $NN \to \gamma' N' N'$

(II) Conformal Transformations in the Momentum Space and Mass Splitting of the Hadrons.

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(I) Analytic and Model-Independent Determination of the Magnetic moments of the Δ Resonances & Deuteron in the πN & NN Bremsstrahlung $\pi N \to \gamma'\pi' N'$ & $NN \to \gamma' N' N'$

CONTENT:

CANCELATION OF THE EXTERNAL AND INTERNAL PARTICLE RADIATION PARTS OF THE BREMSTRALUNG AMPLITUDE

$(A + B \Longrightarrow A' + B' + \gamma')$

METHOD:

$$On\ shell\ Ward - Takahashi\ type\ Identities.$$
$$Current\quad Conservation\quad in\quad (A + B \Longrightarrow A' + B' + \gamma')$$

PUBLICATIONS:

A. I. Machavariani and Amand Faessler. **Journal of Physics G: Nucl. Part. Phys. 37 (2010) 75004.**

A. I. Machavariani and Amand Faessler. **Journal of Physics G: Nucl. Part. Phys. 38 (2011) 035002.**

RESULTS:

Analytic determination of $\gamma\Delta - \Delta'$ vertex through the $\gamma N - N'$, $\gamma\pi - \pi'$ and $\pi N - \Delta$ vertices.

Analytic determination of $\gamma d - d'$ vertex through the $\gamma p - p'$, $\gamma n - n'$ and $np - d$ vertices. CONSEQUENTLY:

$\mu_{\Delta^+} = \frac{M_\Delta}{m_p}\mu_p$ **and** $\mu_{\Delta^{++}} = \frac{3}{2}\mu_{\Delta^+}$

$\mu_{\Delta^o} = \frac{M_\Delta}{m_n}\mu_n$ **and** $\mu_{\Delta^-} = \frac{3}{2}\mu_{\Delta^o}$

$\mu_{deutron}(^3S_1) = \mu_p + \mu_n \quad \mu_{deutron}(^3D_1) \quad Preliminary$

μ_p **- total magnetic moment of the proton.**
μ_n **- total magnetic moment of the neutron**

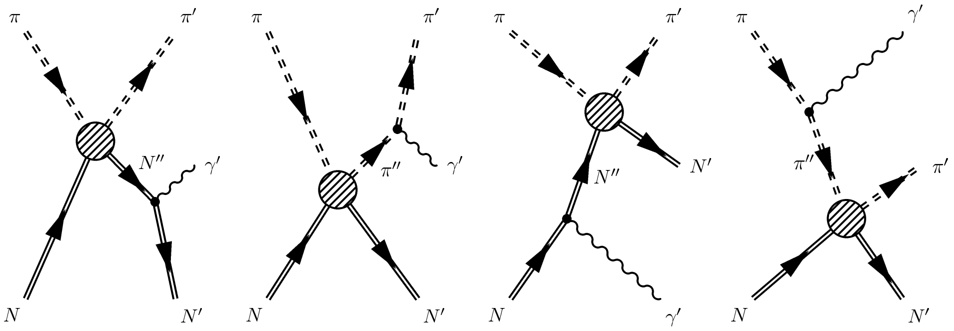

Figure 1: **The external nucleons and pions radiation part $\mathcal{E}^{\mu}_{\gamma'\pi'N'-\pi N}$ ($k'_{\mu}\mathcal{E}^{\mu}_{\gamma'\pi'N'-\pi N} \neq 0!!!$) of the πN bremsstrahlung amplitude. The dashed circle indicates the off shell πN elastic scattering amplitudes.**

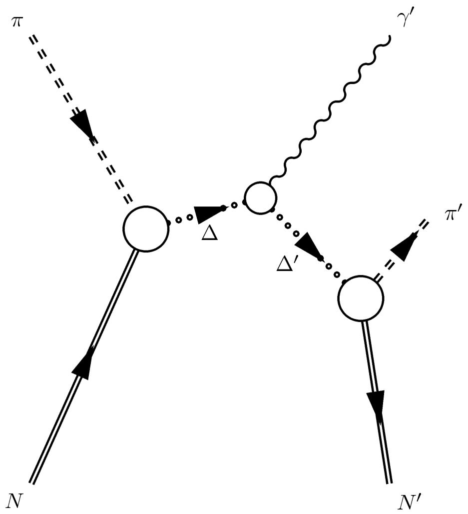

Figure 2: **The double Δ exchange diagram with the intermediate Δ radiation vertex $\mathcal{I}^{\mu}_{\gamma'\pi'N'-\pi N}(\Delta\Delta)$ ($k'_{\mu}\mathcal{I}^{\mu}_{\gamma'\pi'N'-\pi N}(\Delta\Delta) \neq 0!!!$).**

$\Delta - \gamma'\Delta'$ contains the dipole magnetic moment of Δ.

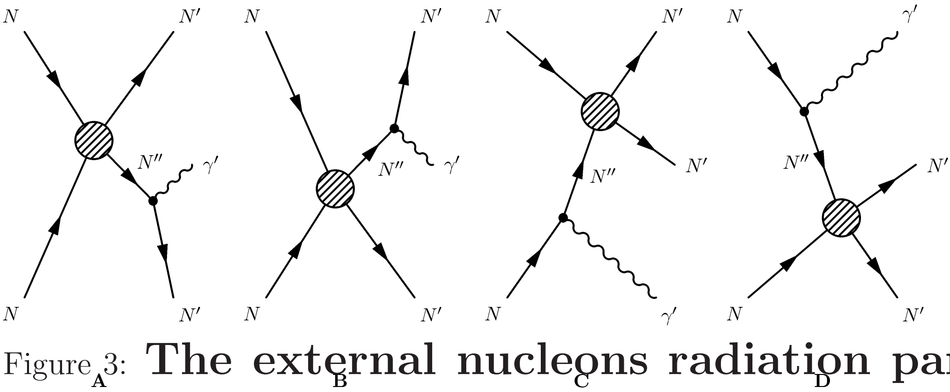

Figure 3: **The external nucleons radiation part $\mathcal{E}^{\mu}_{\gamma' N' N' - NN}$ ($k'_{\mu}\mathcal{E}^{\mu}_{\gamma' N' N' - NN} \neq 0$!!!) of the NN bremsstrahlung amplitude. The dashed circle indicates the off shell NN elastic scattering amplitudes.**

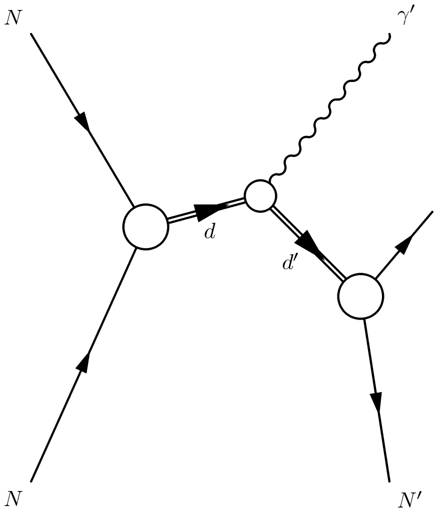

Figure 4: **The double Δ exchange diagram with the intermediate Δ radiation vertex $\mathcal{I}^{\mu}_{\gamma' N' N' - NN}(\gamma' d' - d)$ ($k'_{\mu}\mathcal{I}^{\mu}_{\gamma' N' N' - NN}(\gamma' d' - d) \neq 0$!!!).**
$\gamma' d' - d$ contains the dipole magnetic moment of Δ.

$\gamma' d' - d$ contains the dipole magnetic moment of the deuteron .

$$\mathcal{I}^{3/2}_{\gamma'\pi'N'-\pi N}(\Delta-\gamma\Delta)+(\mathcal{E})^{3/2}_{\gamma'\pi'N'-\pi N}=0,$$

$$\mathcal{I}^{101}_{\gamma'N'N'-NN}(\gamma d-d)+(\mathcal{E})^{101}_{\gamma'N'N'-NN}=0$$

The amplitude of the intermediate Δ or deutron radiation cancel against the special part of the external particle radiation amplitudes!!!

The intermediate Δ or deutron radiation amplitude do not contribute in the observable of the πN or NN bremstrahlung!!!

FORMULATION WITH QUARKS:

BOUND (cluster, composite) STATE approach in Quantum Field Theory

HAAG-NISHIJIMA-ZIMMERMANN (1958) and HUANG-WELDON (1975) approach for the composite particle in quantum field theory.

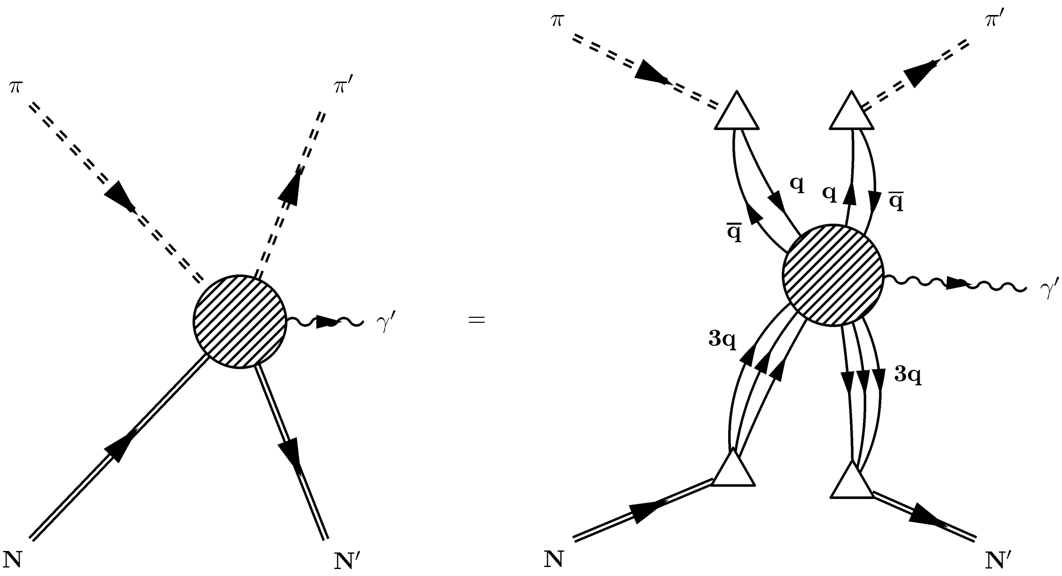

Figure 5: **The πN bremsstrahlung amplitude with the composed nucleons ($3q$) and pions ($q\overline{q}$).**

The commutation or anticommutation rules for the asymptotic STRUCTURELESS particles and the commutation or anticommutation rules for the asymptotic COMPOSITE particles are the same!!!

$\left[\mathsf{a}^{+}_{in(out)}(\mathbf{p'}_{\pi}), \mathsf{a}_{in(out)}(\mathbf{p}_{\pi})\right] = (2\pi)^3 2p^o_{\pi}\delta(\mathbf{p}'_{\pi} - \mathbf{p}_{\pi});$

$\left[\mathsf{a}_{in(out)}(\mathbf{p'}_{\pi}), \mathsf{a}_{in(out)}(\mathbf{p}_{\pi})\right] = \left[\mathsf{a}^{+}_{in(out)}(\mathbf{p'}_{\pi}), \mathsf{a}^{+}_{in(out)}(\mathbf{p}_{\pi})\right] = 0.$

On shell bremsstrahlung amplitudes $< out\pi' N'|J_{\mu}(0)|\pi N; in >$ **as well as** $< out N' N'|J_{\mu}(0)|NN; in >$ **(**$J_{\mu}(x)$ **- photon source), vertices** $< out N'|J_{\mu}(0)|N; in >$**,** $< out\pi'|J_{\mu}(0)|\pi; in >$ **are the same in the formulations with and without quarks.**

THE RESULTING RELATIONS FOR the πN (NN) bremsstrahlung with and without QUARKS ARE THE SAME:

$$\mathcal{I}^{3/2}_{\gamma'\pi' N'-\pi N}(\Delta - \gamma\Delta) + (\mathcal{E}^{3/2})_{\gamma'\pi' N'-\pi N} = 0,$$

$$\mathcal{I}^{101}_{\gamma' N' N'-NN}(\gamma d - d) + (\mathcal{E})^{101}_{\gamma' N' N'-NN} = 0$$

The amplitude of the intermediate Δ or deutron radiation cancel against the special part of the external particle radiation amplitudes, i. e,

the intermediate Δ or deutron radiation amplitude do not contribute in the observable of the πN or NN bremstrahlung!!!

Magnetic moments of the Δ's in units of the nuclear magneton $\mu_N = e/2m_N$.
The index $*$ indicates a theoretical model with the fit of the experimental cross sections of $\pi N \to \gamma'\pi' N'$ for extraction of the magnetic moment μ_Δ.

Model	This work	$SU(6)$ & Bag	Poten. & K-matr.	Skyrme	Low en. pht. theorem	Eff. πN Lagran.	quark
μ_{Δ^+}	3.66	2.79 2.13 2.20-2.45 3.27		2.0-3.0			3.49 2.85 2.3-2.7 2.79
$\mu_{\Delta^{++}}$	5.49	5.5 4.25 4.41-4.89 6.54	6.9-9.7* 4.52±0.95* 5.6-7.5*	4.2-7.4	3.6±2.0* 5.6±2.1* 4.7-6.9* 3.7-4.9*	6.1±0.5*	6.98 5.33 5.1-5.4 6.17
μ_{Δ^o}	-2.504	0. 0. 0. 0.		-1.33-0.19			0. 0.375 -0.3-0.
μ_{Δ^-}	-3.759	-2.79 -2.13 -2.20-2.45 -3.27		-5.62-2.38			-3.49 -2.1 -2.72-3.06

RESULTS (same with and without quarks):

$\mu_{\Delta^+} = \frac{M_\Delta}{m_p}\mu_p$ **and** $\mu_{\Delta^{++}} = \frac{3}{2}\mu_{\Delta^+}$

$\mu_{\Delta^o} = \frac{M_\Delta}{m_n}\mu_n$ **and** $\mu_{\Delta^-} = \frac{3}{2}\mu_{\Delta^o}$

$\mu_{deutron}(^3S_1) = \mu_p + \mu_n \quad \mu_{deutron}(^3D_1) \quad$ *Preliminary*

μ_p **- total magnetic moment of the proton.**
μ_n **- total magnetic moment of the neutron**

CONCLUSION

[I]. Within the usual quantum field theory:

A) The $\gamma\Delta - \Delta$ and $\gamma d - d$ vertices are analytically determined through the $\gamma N - N'$, $\gamma\pi - \pi'$, $\pi N - \Delta$ and $NN - d$ vertices.

B) Analytical expressions for the dipole magnetic moments of the Δ resonances and the deutron are obtained.

[II]. In the present approach was not used any approximations and these results do not dependent on the choice of the models for the $\Delta - \pi N$, $d - NN$, $\gamma\Delta - \Delta$, $\gamma d - d$ and other vertices or effective Lagrangians.

Therefore, the present results are model-independent.

[III]. In the framework of the general field-theoretical approach with and without quark degrees of freedom. are obtained same analytic expressions for μ_Δ and μ_d.

[IV]. It is demonstrated that the intermediate Δ and deutron radiation amplitudes in the $\pi N \to \gamma'\pi'N'$ and $NN \to \gamma'N'N'$ reactions do not contribute in the physical observable, because they cancel completely against the part of the external particle radiation amplitude in the $\pi N \to \gamma'\pi'N'$ and $NN \to \gamma'N'N'$ reactions.

(II) Conformal Transformations in the Momentum Space and Mass Splitting of the Hadrons.

CONTENT:

Conformal transformations in the momentum space in the equivalent 6D, 5D and 4D form within the Dirac model..
Equivalent equation of motion in the 5D and 4D form.

METHOD:

Equivalence of the 6D rotations and the 4D conformal transformations: 4D translations $x'_\mu = x_\mu + a_\mu$, 4D rotations $x'_\mu = \Lambda_{\mu\nu} x^\nu$, scale transformations $x'_\mu = \lambda x_\mu$ and inversion $x'_\mu = -\ell^2 x_\mu / x^2$ (Dirac model).

RESULTS:

Meaning of the fifth dimension.

Equivalent 4D and 5D equations of motion.

Conformal transformations in the momentum space allows to connect two different 4D equation of motion with the same quantum numbers and different masses.

MOST FAMOUS 5D THEORIES:
1. Kaluza-Klein Theory $\sim 1921 - 1926$

Unification of the Einstein general relativity theory and Maxwell electromagnetic theory.

$$g_{AB}(x,x_5) = \begin{pmatrix} g_{\mu\nu} - k^2\Phi^2(x,x_5)A_\mu(x,x_5)A_\nu(x,x_5); & -k\Phi(x,x_5)A_\mu(x,x_5) \\ -k\Phi(x,x_5)A_\nu(x,x_5); & -\Phi^2(x,x_5) \end{pmatrix}$$

where $k = \sqrt{16\pi G/c^4}$**.**
Here both Einstein and Maxwell equations are satisfied! For the fifth dimension variable x_5 **special condition are required.**
RESULTS: Mass and charge spectrum:
ANSATZ:

$$\Phi(x,x_5) = \sum_{n=-\infty}^{\infty} f_n(x)e^{inx_5/\rho}$$

where ρ **is scale parameter**

$$m_n = |n|/\rho$$

$$e_n = n\frac{4\sqrt{\pi G}}{\rho}$$

and elementary charge is $e = \frac{4\sqrt{\pi G}}{\rho}$**! QUANTIZATION of charge is achieved.**
Modern formulations: chiral fermions (Witten), Supersymmetries, cosmology, gravity etc.
Meaning of the fifth dimension — nonobserved (internal) dimension which generates the auxiliary fields;

MOST FAMOUS 5D THEORIES: Proper time method of Fock-Schwinger (1938;1951):

$$\tau^2 = x_0^2 - \vec{x}^2$$

fifth dimension — proper time (auxiliary dimension and auxiliary fields; Feynman,Nambu Stueckelberg, Barut etc)

Mass variable as the conjugate to τ. mass quantization; Greenberg

$$m^2 = p_0^2 - \vec{p}^2$$

APPLICATION: Particle in the constant external field.

MOST FAMOUS 5D THEORIES:

Fundamental mass (Length) theory: Heisenberg (1939-1953), Markov (1961-1966), Kadyshevsky (1961-1985)

FUNDAMENTAL CONSTANTS: c.g.s.

speed of light	**c**	ŁT^{-1}	**3.0**$\times 10^{10}$
Gravitational constant	**G**	$M^{-1}L^3T^{-2}$	**6.7**$\times 10^{-8}$
Planck's constant	**h**	**M**$T^{-1}L^2$	**6.6**$\times 10^{-27}$
electron charge	**e**	$M^{1/2}L^{3/2}T^{-1}$	**4.8**$\times 10^{-10}$

$$m_{PL} = \frac{hc}{G} = 2.2 \times 10^{-5} g$$

$$l_{PL} = \frac{Gh}{c^3} = 1.6 \times 10^{-33} cm$$

$$t_{PL} = \frac{Gh}{c^5} = 5.4 \times 10^{-44} s$$

m_{PL} **is to big**
Fifth dimension — auxiliary fields;

MOST FAMOUS 5D THEORIES:

1. Conformal group (Bateman and Cunningham 1910; Dirac 1936; Mack and Salam 1969,...)

Conformal group of transformations of the point-like object in 4D space

$$TRANSLATION: \qquad x'_\mu = x_\mu + a_\mu$$

(homogeneity of space-time $\Longrightarrow$ energy-momentum conservation conditions)

$$ROTATIONS: \qquad x'_\mu = \Lambda_{\mu\nu} x^\nu$$

(homogeneity of space-time $\Longrightarrow$ 3D angular-momentum conservation)

$$SCALE \quad TRANSFORMATION: \qquad x'_\mu = e^\alpha x_\mu$$

which destroy the Poincare-invariance of the particle states with nonzero mass

$$x^\mu \frac{d}{dx^\mu} \; is \; scale \; invariant \;\; \longrightarrow \;\; but \; p'_\mu \neq e^{-\alpha} p_\mu$$

because

$$p \equiv (\sqrt{m^2 + p^2}, \mathbf{p} \qquad \longrightarrow \qquad p'_\mu \neq e^{-\alpha} p_\mu, \quad m'^2 = e^{-2\alpha} m^2$$

$INVERSION: \qquad x_\mu = -l^2 x_\mu / x^2$

INVERSION $\Longrightarrow$ NONLINEAR TRANSFORMATION

EXAMPLES OF INVERSION

1. Numbers of Fibonacci - (sun of the good-natured man!)

1202 Leonardo Fibonacci from Pizza: boundary conditions

1. **Reproduction of rabbits is started from two months !**
2. **Every pair of rabbits produces new pair of rabbits.**
3. **Rabbits have great, infinity vitality. They never die!**

months$\equiv n$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2rabbits$\equiv f_n$	**0**	**1**	**1**	**2**	**3**	**5**	**8**	**13**	**21**	**34**	**55**
f_{n+1}/f_n	∞	**1**	**2**	**1.5**	**1.6**	**1.6**	**1.625**	**1.615**	**1.619**	**1.6176**	**1.6182**

Boundary conditions: $f_o = 0 \quad f_1 = 1$ **SEQUENCE:** $f_{k+2} = f_k + f_{k+1}$

: **R. Simpson 1780 RATIO** $\tau = \lim_{n\to\infty} = f_{n+1}/f_n$

GOLD SECTION $\Longrightarrow \tau - 1/\tau = 1 \Longleftarrow$ **1D INVERSION SYMMETRY (Hamiltonian of Todorov: H=h-1/h)**

$\tau = 2cos(\pi/5) \qquad f_n = \frac{\tau^n - (-\tau)^n}{\sqrt{5}}$

Inversion as transformation between the internal and external domains

$x_{inr} = 1/x_{ext}$ $\quad$ $x_{ext} = 1/x_{inr}$

0 $\quad \{x_{inr}\}$ $\quad$ **1** $\quad \{x_{ext}\}$

CONFORMAL GROUP:
Maximal space-time (Geometric) transformations of the point-like object (without structure) in the physical 4D flat space!

Conformal symmetry is the MAXIMAL Symmetry of the Maxwell equations (Bateman and Canninghem 1910!)

Conformal group as the rotation group SO(2,4) in the configuration space.

Translation+Rotations+Scale transformations+Inversions in the 4D space $\Longleftrightarrow$ rotations in the 6D space on the 6D cone

$$6D - cone(INVARIANT): \qquad \xi_\mu \xi^\mu + \xi_5 \xi^5 - \xi_6 \xi^6 = 0,$$

$$4D - variable: \qquad x_\mu = \frac{\xi_\mu}{\xi_+}; \qquad \xi_\pm = \frac{\xi_5 \pm \xi_6}{l}.$$

$TRANSLATION: \quad x'_\mu = x_\mu + a_\mu \Longleftrightarrow$ **ROTATION in the** $(\mu, 5)$ **and** $(\mu, 6)$ **planes**

$\xi'_\mu = \xi_\mu + a_\mu \xi_+ \quad and \quad \xi'_+ = \xi_+; \quad \xi'_- = \xi_- + 2a_\nu \xi^\nu / l + a^2 \xi_+ / l$**,**

$4D - ROTATION: \quad x'_\mu = \Lambda_{\mu\nu} x^\nu \Longleftrightarrow$ **ROTATION in the** (μ, ν) **planes**

$\xi'_\mu = \Lambda_{\mu\nu} \xi^\nu$

$Scale\ transformations: \quad x'_\mu = e^{-\lambda} x_\mu \Longleftrightarrow$ **ROTATION in the** $(5, 6)$ **plane** $\xi'_\mu = \xi_\mu \quad \xi'_\pm = e^{\pm\lambda} \xi_\pm$

$i.\ e. \quad \xi'_5 = ch\lambda \xi_5 + sh\lambda \xi_6 \quad \xi'_6 = sh\lambda \xi_5 + ch\lambda \xi_6$

$Inversion: \quad x_\mu = -l^2 x_\mu / x^2 \Longleftrightarrow$ **ROTATION in the** $(5, 6)$ **plane** $\xi'_\mu = \xi_\mu \quad \xi'_\pm = \xi_\mp$

$i.\ e. \quad \xi'_5 = \xi_5 \quad \xi'_6 = -\xi_6 \quad$ **reflection of the 6-th axes**

Meaning of the fifth dimension — auxiliary (mathematical) dimension;

Conformal group as the rotation group SO(2,4) in the momentum space.

Translation+Rotations+Scale transformations+Inversions in the 4D momentum space $\Longleftrightarrow$ rotations in the 6D space on the 6D cone

$$6D - cone(INVARIANT): \qquad \kappa_\mu\kappa^\mu + \kappa_5\kappa^5 - \kappa_6\kappa^6 = 0,$$

$$4D - variable: \qquad q_\mu = \frac{\kappa_\mu}{\kappa_+}; \qquad \kappa_\pm = \frac{\kappa_5 \pm \kappa_6}{M}.$$

$$M \qquad scaleparameter$$

$TRANSLATION: \qquad q'_\mu = q_\mu + h_\mu \Longleftrightarrow$ **ROTATION in the** $(\mu, 5)$ **and** $(\mu, 6)$ **planes**
$\kappa'_\mu = \kappa_\mu + h_\mu\kappa_+ \quad and \quad \kappa'_+ = \kappa_+; \quad \kappa'_- = \kappa_- + 2h_\nu\kappa^\nu/M + h^2\kappa_+/M$**,**

$4D - ROTATION: \qquad q'_\mu = \Lambda_{\mu\nu}q^\nu \Longleftrightarrow$ **ROTATION in the** (μ, ν) **planes**
$\kappa'_\mu = \Lambda_{\mu\nu}\kappa^\nu$

$Scale\ transformations: \qquad q'_\mu = e^{-\lambda}q_\mu \Longleftrightarrow$ **ROTATION in the** $(5, 6)$ **plane** $\kappa'_\mu = \kappa_\mu \quad \kappa'_\pm = e^{\pm\lambda}\kappa_\pm$
$i.\ e. \quad \kappa'_5 = ch\lambda\kappa_5 + sh\lambda\kappa_6 \quad \kappa'_6 = sh\lambda\kappa_5 + ch\lambda\kappa_6$

$Inversion: \qquad q_\mu = -M^2 q_\mu/q^2 \Longleftrightarrow$ **ROTATION in the** $(5, 6)$ **plane** $\kappa'_\mu = \kappa_\mu \quad \kappa'_\pm = \kappa_\mp$
$i.\ e. \quad \kappa'_5 = \kappa_5 \quad \kappa'_6 = -\kappa_6$ **reflection of the 6-th axes.**

Conformal group of the transformations in the 4D momentum (OFF-MASS SHELL) flat space

$$4D - TRANSLATION: \qquad q'_\mu = p_\mu + h_\mu$$

(TRANSLATION INVARIANCE $\Longleftrightarrow$ homogeneity of the 4D momentum space $\Longrightarrow$ gauge transformation with the constant $A_\mu(x) = h_\mu$; gradient invariance!)

$$p'_\mu = p_\mu + h_\mu \Longrightarrow i\frac{\partial}{\partial x^\mu} + A_\mu(x) \quad A_\mu(x) = h_\mu$$

Conformal group of the transformations in the 4D momentum (OFF-MASS SHELL) flat space

INVARIANT under 6D ROTATIONS

$$\kappa_A\kappa^A \equiv \kappa_\mu\kappa^\mu + \kappa_5^2 - \kappa_6^2 = 0,$$

q_μ $(\mu = 0,1,2,3)$ **off shell four-momentum** $q_o \neq \sqrt{m^2 + q_i q_i}$

$$q_\mu = M\kappa_\mu/(\kappa_5+\kappa_6) \equiv \kappa_\mu/\kappa_+; \quad M \ \textit{is a scaleparameter.}$$

The 6D cone $\kappa_A\kappa^A = 0$ is equivalent to

$$q_\mu q^\mu + M^2\frac{\kappa_-}{\kappa_+} = 0, \quad with \quad \kappa_\pm = \frac{\kappa_5 \pm \kappa_6}{M}$$

Condition: $\phi(\kappa)$ **is on** $\kappa_A\kappa^A = 0$**:**

$$\Big(\frac{\partial^2}{\partial\xi^\mu\partial\xi_\mu}+\frac{\partial^2}{\partial\xi^5\partial\xi_5}-\frac{\partial^2}{\partial\xi^6\partial\xi_6}\Big)\int d^6\kappa e^{i\kappa_A\xi^A}\delta(\kappa_\mu\kappa^\mu+\kappa_5^2-\kappa_6^2)\phi(\kappa) = 0.$$

5D-reduction of $\kappa_A\kappa^A = 0$ to cover $-\infty < q^2 < +\infty$

$$q_\mu q^\mu + q_5^2 = M^2 \quad with \quad \frac{q_5^2}{M^2} = \frac{\kappa_5-\kappa_6}{\kappa_5+\kappa_6}+1,$$

$$q_\mu q^\mu - q_5^2 = -M^2 \quad with \quad \frac{q_5^2}{M^2} = -\frac{\kappa_5-\kappa_6}{\kappa_5+\kappa_6}+1.$$

Conditions: $\varphi(x,x_5)$ **are on** $q^2 \pm q_5^2 = \pm M^2 = 0$**:**

$$\Big(\frac{\partial^2}{\partial x^\mu\partial x_\mu}+\frac{\partial^2}{\partial x^5\partial x_5}+M^2\Big)\varphi_1(x,x_5) = 0$$

$$\Big(\frac{\partial^2}{\partial x^\mu\partial x_\mu}-\frac{\partial^2}{\partial x^5\partial x_5}-M^2\Big)\varphi_2(x,x_5) = 0.$$

$$\varphi(x,x_5)=\varphi_1(x,x_5)+\varphi_2(x,x_5),$$

$$\frac{\partial^2\varphi(x,x_5)}{\partial x^\mu\partial x_\mu}+(\frac{\partial^2}{\partial x^5\partial x_5}+M^2)\varphi_-(x,x_5)=0$$

$$\frac{\partial^2\varphi_-(x,x_5)}{\partial x^\mu\partial x_\mu}+(\frac{\partial^2}{\partial x^5\partial x_5}+M^2)\varphi(x,x_5)=0$$

It is necessary to introduce

$$\varphi_-(x,x_5)=\varphi_1(x,x_5)-\varphi_2(x,x_5).$$

5D equation of motion

$$\Big(\frac{\partial^2}{\partial x^\mu\partial x_\mu}+m^2\Big)\varphi(x,x_5)=j_+(x,x_5)$$

$$\Big(\frac{\partial^2}{\partial x^\mu\partial x_\mu}+m_-^2\Big)\varphi_-(x,x_5)=j_-(x,x_5)$$

4D equation of motion

$$\Big(\frac{\partial^2}{\partial x^\mu\partial x_\mu}+m^2\Big)\Phi_\pm(x)=J_\pm(x)$$

$$\Phi_\pm(x)=\varphi_\pm(x,x_5=0)\quad J_\pm(x)=j_\pm(x,x_5=0)$$

Boundary condition over 5-th coordinate

$$M^2\Big[1+(\frac{1}{M}\frac{\partial}{\partial x_5})^2\Big]\varphi_-=m^2\varphi-j_+;\quad M^2\Big[1+(\frac{1}{M}\frac{\partial}{\partial x_5})^2\Big]\varphi=m_-^2\varphi_--j_-.$$

RESULTS:

Meaning of the fifth dimension.

General scheme of introduction 5-th dimension through Conformal transformations.

Other 5D formulations (Kaluza-Klein, Fock-Schwinger, ...) can be incorporated

Equivalent 4D and 5D equations of motion.

Conformal transformations in the momentum space allows to connect two different 4D equation of motion with the same quantum numbers and different masses.

electron—μ-meson,
Pion(139)—-Pion(1300)
nucleon(939)—-N' etc