Strong coupling constant from τ decay within a dispersive approach to QCD

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The strong coupling constant α_s, at some conventionally chosen scale, is one of the fundamental parameters within Standard Model. Precise values for α_s are important for high-precision tests of the Standard Model.

One of the most precise determinations of α_s (at the low energy scale of the τ lepton mass $m_\tau = 1.777\,GeV$) comes from hadronic tau decay: See Seminal work:

Braaten, E., Narison, S., and Pich, A. Nucl. Phys.(1992)

The inclusive semi-hadronic decay rate

$$R_\tau = \Gamma(\tau^- \to hadrons\,\nu_\tau(\gamma))/\Gamma(\tau^- \to e^-\bar{\nu}_e\nu_\tau(\gamma))) \tag{1}$$

is very sensitive to the precise value of α_s.

An independent highest-precision determination of the coupling at low energies comes from a lattice perturbation theory analysis of UV-sensitive lattice observables. These two highest-precision determinations extrapolated to the Z mass yield

$$\alpha_s(M_z^2) = 0.1212 \pm 0.0011 \quad (\tau \text{ decay M. Davier et al.}) \quad (2)$$

$$\alpha_s(M_z^2) = 0.1170 \pm 0.0012 \quad (\text{lattice Q. Masson etal.}). \quad (3)$$

Note that the agreement between these two results, with the errors quoted, is not good. They differ from each other by about 2.6 standard deviations. Furthermore, the lattice determination is closer to $\alpha_s(M_z^2)$ values obtained from high energy experiments. Thus, the reliability of the estimates from the τ-lepton data has been called in question.

Theoretical problems

● The notion of **quark-hadron duality** and its possible violation: PT does not work locally in Minkowski space even at large energies because of non-perturbative color confinement effects. Fortunately, owing the idea of quark-hadron duality

C. Poggio H. Quinn and S. Weinberg 1976

one may apply the quark-gluon perturbation theory (PT), but only for inclusive quantities like decay rates.

● The basic principle is **analyticity** from which one derives the finite energy sum rules relating the decay rate with the integral of current-current correlation function.

$$\Pi_{\mu\nu,ij}^{V/A}(p) = (p_\mu p_\nu - g_{\mu\nu})\Pi^{V/A,(1)}(p) + p_\mu p_\nu \Pi^{V/A,(0)}(p) \tag{4}$$

$$\Pi^{V/A}_{\mu\nu,ij}(p) = \imath \int e^{\imath p.x} < \Omega\, T\{J^{V/A}_{\mu,ij}(x) J^{V/A}_{\nu,ij}(0)^{+}\}\Omega > \qquad (5)$$

$J^{V}_{\mu,ij}(x) = \bar{\psi}_j(x)\gamma_\mu\psi_i(x)$ and $J^{A}_{\mu,ij}(x) = \bar{\psi}_j(x)\gamma_\mu\gamma_5\psi_i(x)$ with $i,j = u,d,s$ The last quantity is calculated in PT.

The hadronic spectral functions

$$v_1(s)/a_1(s) = 2\pi Im \Pi^{(1)}_{ud,v/A}(s)$$

The decay rate

$$R_\tau = 12\pi \int_0^{m_\tau^2} \frac{d\,s}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2 \left[(1 + 2\frac{s}{m_\tau^2})(v_1(s) + a_1(s))/2\pi + a_0(s)/2\pi\right] \qquad (6)$$

Since

$$\Pi^{(J)}(s) = |V_{ud}|^2[\Pi^{(V,J)}_{ud}(s) + \Pi^{(A,J)}_{ud}(s)] + |V_{us}|^2[\Pi^{(V,J)}_{us}(s) + \Pi^{(A,J)}_{us}(s)]$$

with V_{ij} being the elements of the Cabibbo-Kobayashi-Maskawa quark mixing matrix elements,

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,s}$$

It is convenient to rewrite the decay rate in terms of physical quantity, the Adler function

$$D^{(1+0)}(s) = -s \frac{d}{d\,s} \Pi^{(1+0)}(s) \tag{7}$$

Main theoretical tool to calculate the correlator in QCD is the renormalization group improved PT supplemented with the Wilson operator product expansion (OPE).

■ The renormalization group (RG) invariance cannot be used unambiguously in the time-like region.

● The RG improvement and the analytical continuation procedures in finite order perturbation theory do not commute. For this reason there are two approaches to improve the perturbative expansions with the help of the RG, namely fixed-order and contour improved perturbation theory the FOPT and CIPT respectively. Within FOPT extracted numerical values of α_s have been always lower.

● The RG improved approximations to the current-current correlation functions parameterized in terms of the running coupling do not obey correct analytical properties of the corresponding exact quantities. The analytical properties are violated due to the non-physical Landau singularities of the perturbative running coupling that appear at small space-like momenta. This destroy dispersion relation for the correlator which is crucial in the analysis.

Supposedly, these singularities may deteriorate the extracted values of α_s within CIPT and FOPT .

● Dispersive Approaches (DA) are free from this obstacle. Within DA the Landau singularities are systematically removed from the observables order by order in PT.

Analytic Perturbation Theory a simple and effective dispersive technique (APT)

D. Shirkov, I. Solovtsov, (1998,2000,2006)
I. Solovtsov, O. Solovtsova, K. Milton (1999)

The τ lepton decay rate has been analyzed within (APT)

Milton, K.A., Solovtsov, I.L., Solovtsova, O.P., Yasnov, V.I. (2000)

However, APT predict , from the non-strange τ lepton decay data, too large value for the strong coupling constant, $\alpha_s(m_\tau^2) = 0.403 \pm 0.015$ which can not be accepted.

● The QCD perturbation theory supplemented with the **OPE** fail to describe the detailed infrared behavior of the Adler function associated with the τ decay rate. This shortcoming remains in APT too.

●In this paper we suggest a more sophisticated dispersive approach to the τ decay which is based on the approximation to the Adler function with correct IR and UV properties. Part of the results are published in

Magradze, Few-Body Syst. (2010) 48, 143-169

Our starting point is the semi-empirical representation for the vector spectral function

$$v_1^{\text{“semi−emp.”}}(s) = \theta(s_{\mathrm{p}} - s) v_1^{\text{exp.}}(s) + \theta(s - s_{\mathrm{p}}) v_1^{\text{pQCD}}(s), \quad (8)$$

where s_{p} is the onset of perturbative continuum, an infrared boundary in Minkowski region above which we trust pQCD.

$$0 < s_p < m_\tau^2$$

$v_1^{\text{exp.}}(s)$ is measured with high precision by ALEPH and OPAL collaborations in the range $0 < \sqrt{s} < m_\tau = 1.777\, GeV$.

Schael, S. et al.: [ALEPH Collaboration]. Phys. Rept. **421**, 191 (2005)

$v_1^{\text{pQCD}}(s)$ is the perturbation theory approximation to the spectral function, in what follows we show that

$$v_1^{\text{pQCD}}(s) = v_1^{\text{APT}}(s)$$

Previously analogical representation used by

Bertlmann, R.A., Launer, G. and deRafael,
Peris, S., Perrottet, M., de Rafael, E. (1998).

In this formulation one does not rely on the procedure of the analytical continuation of the truncated OPE to Minkowski region, a source of the possible DVs. We assume that, the non-perturbative description is essential only in the low energy region $0 < s < s_{\rm p}$. But, in this region we will use the measured on the experiment spectral function.

Our aim is to utilize the total information encoded in formula (8). We will use it to extract the value of α_s from the τ data.

The Adler function, the object determined in the space-like region ($q^2 = -Q^2$ and $Q^2 > 0$ for space-like momenta)

$$D(Q^2) = Q^2 \int_0^\infty \frac{2v_1(s)ds}{(s+Q^2)^2}, \qquad (9)$$

The “semi-experimental” Adler function is obtained by inserting *ansatz* (8) into integral (9)

$$D_{\text{“semi-exp.”}}(Q^2) = D_{\text{exp.}}(Q^2, s_{\text{p}}) + D_{\text{pQCD}}(Q^2, s_{\text{p}}), \tag{10}$$

where the experimental and perturbation theory components of the total “semi-experimental” Adler function are defined as

$$D_{\text{exp.}}(Q^2, s_{\text{p}}) = Q^2 \int_0^{s_{\text{p}}} \frac{2v_1^{\text{exp.}}(s)\,d\,s}{(s+Q^2)^2}, \tag{11}$$

$$D_{\text{pQCD}}(Q^2, s_{\text{p}}) = Q^2 \int_{s_{\text{p}}}^{\infty} \frac{2v_1^{\text{pQCD}}(s)\,d\,s}{(s+Q^2)^2} \tag{12}$$

Note that the “semi-experimental” Adler function is not wholly experimental quantity, since it depends also on the theoretical component $D_{\text{pQCD}}(Q^2, s_{\text{p}})$.

Theoretical Framework

The main quantity of interest for following analysis is the Adler function associated with the vector current two-point correlator. The perturbative expansion of this function in the limit of vanishing quark masses reads

$$D_{PT}(Q^2) = \sum_{n=0}^{\infty} a_s^n(\mu^2) \sum_{k=1}^{n+1} k c_{n,k} L^{k-1} \quad \text{where} \quad L \equiv \ln \frac{Q^2}{\mu^2}, \tag{13}$$

$a_s(\mu^2) = \frac{\alpha_s(\mu^2)}{\pi}$ with $\alpha_s(\mu^2)$ being the strong coupling constant renormalized at the scale μ. Using RI the expansion (13) may be reexpressed as an asymptotic expansion in powers of the running coupling $\alpha_s(Q^2)$

$$D_{\mathrm{RGI}}(Q^2) = \sum_{k=0}^{\infty} d_k \left(\frac{\alpha_s(Q^2)}{\pi} \right)^k, \tag{14}$$

where $d_n = c_{n,1}$ and the subscript "RGI" refers to the renormalization group improved perturbation theory. The first two coefficients in series (14) are universal $d_0 = d_1 = 1$. In the $\overline{\text{MS}}$ scheme for $n_f = 3$ quark flavours the known coefficients take values

$$d_2 \simeq 1.6398,\ d_3 \simeq 6.3710\ d_4 \simeq 49.0757.$$

the last coefficient d_4 in the case of massless quarks has been calculated recently by using powerful computational techniques:

Baikov, P.A., Chetyrkin, K.G., Kühn, J.H.:(2008)

The exact Adler function $D(z)$ ($z = Q^2 = -q^2$) is known to be analytic except the cut running along the negative real axis.

This fact enables us to calculate the hadronic non-strange vector spectral function from the Adler function via the contour integral

$$v_1(s) = \frac{1}{4\pi\imath} \oint_{-s-\imath\epsilon}^{-s+\imath\epsilon} \frac{D(z)}{z} d\,z, \tag{15}$$

where the path of integration, connecting the points $-s \mp \imath\epsilon$ on the complex z-plane, avoids the cut running along the real negative axis. We shall assume, without loss of generality, that the approximation (14) to the Adler function has only one unphysical singularity located on the positive real axis. This is the case, for example, if we use the exact (explicitly solved) two-loop order running coupling in $\overline{\text{MS}}$ like renormalization schemes

B. Magradze "QUARKS-98" (1998)
Gardi, E., Grunberg, G., Karliner, M.:(1998)J. High Energy Phys.

07, 007 (1998)

The explicit expression for the $\overline{\text{MS}}$ scheme running coupling at the two-loop order reads

$$a_s^{(2)}(Q^2) = -\frac{\beta_0}{\beta_1}\frac{1}{1+W_{-1}(\zeta)}: \quad \zeta = -\frac{1}{eb_1}\left(\frac{Q^2}{\Lambda^2}\right)^{-1/b_1}, \qquad (16)$$

where β_0 and β_1 are the first two β-function coefficients

$$\beta_0 = \frac{1}{4}\left(11 - \frac{2}{3}n_f\right), \quad \beta_1 = \frac{1}{16}\left(102 - \frac{38}{3}n_f\right),$$

$b_1 = \beta_1/\beta_0^2$, $\Lambda \equiv \Lambda_{\overline{\text{MS}}}$ and W_{-1} denotes the branch of the Lambert W function. On the other hand, a running coupling at higher orders may be expanded in powers of the exact (explicitly solved) two-loop order coupling

Kourashev, D.S., Magradze, B.A.: Theor. Math. Phys. **135**, 531 (2003)

$$\alpha_s^{(\mathrm{k-loops})}(Q^2) = \sum_{n=1}^{\infty} \mathcal{C}_n^{(k)} \alpha_s^{(\mathrm{two-loops})n}(Q^2)|_{\mathrm{exact}}, \qquad (17)$$

where the numerical coefficients $\mathcal{C}_n^{(k)}$ are determined in terms of the β-function coefficients (see Appendix A). It has been shown in

Magradze, B.A.: Few-Body Systems **40**,71-99 (2006)

that this series has a sufficiently large radius of convergence in the space of the coupling constants, and its partial sums provide very accurate approximations to the exact k-th order ($k > 2$) coupling in the complex Q^2 plane. The Adler function evaluated with this approximation to the coupling has an unphysical singularity located on the positive Q^2-axis. The corresponding cut runs along the finite interval of the positive Q^2-axis. Nevertheless, formula (15) is still valid provided that the integration contour avoids both the physical and unphysical cuts.

Let us separate out the parton level term from the perturbation theory approximation to the Adler function

$$D_{\mathrm{RGI}}(Q^2) = 1 + d_{\mathrm{RGI}}(Q^2): \quad d_{\mathrm{RGI}}(Q^2) = \sum_{k=1}^{\infty} d_k a_s^k(Q^2), \quad (18)$$

where $a_s(Q^2) = \alpha_s(Q^2)/\pi$. As it was discussed above, the function $d_{\mathrm{RGI}}(Q^2)$ is analytic except the cuts running along the real

Q^2-axis. The physical cut runs along the real negative interval $-\infty < Q^2 < 0$, and the unphysical cut runs along the positive interval $0 < Q^2 < s_{\mathrm{L}}$, where the point $Q^2 = Q^2_{\mathrm{L}} \equiv s_{\mathrm{L}} > 0$ corresponds to the "Landau singularity". We may then write a Cauchy relation

$$d_{\mathrm{RGI}}(Q^2) = \frac{1}{2\pi\imath} \oint_{\Gamma} \frac{d_{\mathrm{RGI}}(w)}{w - Q^2} d\,w \tag{19}$$

where the integral is taken round the closed contour Γ drawn in Fig.1. We easily derive a violated dispersion relation (DR) for the function $d_{\mathrm{RGI}}(Q^2)$, using the asymptotic condition $d_{\mathrm{RGI}}(z) \to 0$ as $|z| \to \infty$,

$$d_{\mathrm{RGI}}(Q^2) = d_{\mathrm{APT}}(Q^2) + d_{\mathrm{L}}(Q^2) \tag{20}$$

where the function $d_{\mathrm{APT}}(Q^2)$ satisfies the normal DR

$$d_{\mathrm{APT}}(Q^2) = \frac{1}{\pi} \int_0^{\infty} \frac{\rho_{\mathrm{eff}}(\sigma)}{\sigma + Q^2} d\,\sigma, \tag{21}$$

with the effective spectral density

$$\rho_{\text{eff}}(\sigma) = \text{Im}\{d_{\text{RGI}}(-\sigma - \imath 0)\}. \tag{22}$$

It is to be noted here that the function

$$D_{\text{APT}}(Q^2) = 1 + d_{\text{APT}}(Q^2) \tag{23}$$

is the analytic image of the perturbative Adler function determined in the sense of the Analytic Perturbation Theory (APT) approach of Shirkov and Solovtsov. The second term in (20), which violates the DR, corresponds to the contribution to the integral (19) coming from the "Landau branch cut". It is represented by the contour integral

$$d_{\text{L}}(Q^2) = -\frac{1}{2\pi\imath}\oint_{C_L^+} \frac{d_{\text{RGI}}(\zeta)}{\zeta - Q^2} d\,\zeta, \tag{24}$$

taken round the circle $\{\zeta : \zeta = s_{\mathsf{L}} + s_{\mathsf{L}} \exp{(\imath\phi)}, -\pi < \phi \leq \pi\}$ in the positive (anti-clockwise) direction. s_{L} corresponds to the Landau singularity.

The perturbation theory approximation to the hadronic spectral function is calculated by inserting the series (18) into the inversion formula (15). An important point is that the "Landau part" $d_{\mathsf{L}}(Q^2)$ does not contribute into the spectral function, provided that $s > 0$. Thus one finds the expression for the hadronic spectral function in terms of the effective spectral density

$$v_1^{\mathsf{pQCD}}(s) \equiv v_1^{\mathsf{APT}}(s) = \frac{1}{2}(1 + r(s)), \tag{25}$$

where

$$r(s) = \frac{1}{\pi}\int_s^{\infty} \frac{\rho_{\mathsf{eff}}(\sigma)}{\sigma} d\,\sigma. \tag{26}$$

With the help of formula (25), we express the "perturbative component" of the total "experimental" Adler function in terms of the effective spectral density

$$D_{\mathrm{pQCD}}(Q^2, s_p) = \int_{s_p}^{\infty} \mathcal{K}(Q^2, s)(1 + r(s))d\, s \tag{27}$$

where we have introduced the notation $\mathcal{K}(Q^2, s) = Q^2/(s+Q^2)^2$. Integrating (27) by parts we obtain a more convenient representation

$$D_{\mathrm{pQCD}}(Q^2, s_p) = \frac{Q^2}{s_p + Q^2}(1+r(s_p)) - \frac{Q^2}{\pi}\int_{s_p}^{\infty} \frac{\rho_{\mathrm{eff}}(\sigma)}{\sigma(\sigma + Q^2)}d\, \sigma. \tag{28}$$

● The power suppressed part of the total “semi-experimental” Adler function is determined as

$$D_{\text{pow.sup.}}(Q^2, s_{\text{p}}) = D_{\text{“semi-exp.”}}(Q^2) - D_{\text{RGI}}(Q^2). \qquad (29)$$

Combining formulas (10), (20), we rewrite formula (29) in the form

$$\begin{aligned} D_{\text{pow.sup.}}(Q^2, s_{\text{p}}) &= D_{\text{exp.}}(Q^2, s_{\text{p}}) + D_{\text{pQCD}}(Q^2, s_{\text{p}}) - D_{\text{RGI}}(Q^2) \\ &= \int_0^{s_{\text{p}}} K(Q^2, s) 2v_1^{\text{exp.}}(s) d\, s - d_{\text{L}}(Q^2) - \int_0^{s_{\text{p}}} K(Q^2, s) 2v_1^{\text{APT}}(s) d\, s. \end{aligned} \qquad (30)$$

From definitions (24) and (27), we obtain the asymptotic formulas

$$\mathcal{K}(Q^2, s) \approx Q^{-2} + \mathcal{O}(sQ^{-4}), \quad d_{\text{L}}(Q^2) \approx c_{\text{L}} \Lambda^2 Q^{-2} + \mathcal{O}(\Lambda^4 Q^{-4}) \qquad (31)$$

as $Q^2 \to \infty$ where $\Lambda \equiv \Lambda_{\overline{\text{MS}}}$, the conventional $\overline{\text{MS}}$-scheme QCD parameter. Since the parameter s_{L} is proportional to Λ^2 the coefficient c_{L} is a positive number independent of Λ

$$c_{\mathsf{L}} = \Lambda^{-2}\frac{1}{2\pi\imath}\oint_{\mathsf{C}_{\mathsf{L}}^{+}} d_{\mathsf{RGI}}(\zeta)d\,\zeta = \frac{1}{2\pi}\frac{s_{\mathsf{L}}}{\Lambda^2}\int_{-\pi}^{\pi} d_{\mathsf{RGI}}(s_{\mathsf{L}} + s_{\mathsf{L}}e^{\imath\phi})d\,\phi \quad (32)$$

Using formulas (30) and (31), we write asymptotic expansion for $D_{\mathsf{pow.sup.}}(Q^2, s_p)$. It follows from the OPE that the leading term proportional to Q^{-2} in the asymptotic expansion vanishes if the quarks are massless. This leads to the equation

$$c_{\mathsf{L}}\Lambda^2 + s_{\mathsf{p}}(1 + r(s_{\mathsf{p}})) + \frac{1}{\pi}\int_0^{s_{\mathsf{p}}} \rho_{\mathsf{eff}}(\sigma)d\,\sigma = \int_0^{s_{\mathsf{p}}} 2v_1^{\mathsf{exp.}}(s)d\,s. \quad (33)$$

To calculate coefficient c_{L} numerically we use formula (32) and exact (numeric) four-loop running coupling The values of c_{L} obtained in the $\overline{\mathsf{MS}}$ scheme are listed in Table 1. In the calculations we have used the approximations to the Adler function of increasing order. For the unknown $\mathcal{O}(\alpha_s^5)$ correction to the Adler function, we use the geometric estimate $d_5 = d_4(d_4/d_3) = 378$.

	The QCD correction to the Adler function				
	LO	NLO	N^2LO	N^3LO	N^4LO
c_L	0.301262	0.453421	0.555401	0.651373	0.721687

Tab. 1

It follows from the semi-empirical representation (8) for the spectral function that one may calculate in perturbation theory the decay rate of the τ lepton into hadrons of invariant mass larger than $\sqrt{s_\mathrm{p}}$

$$R_{\tau,V}^{\mathrm{pert.}}|_{s>s_\mathrm{p}} = 6|V_{ud}|^2 S_{EW} \int_{s_\mathrm{p}}^{m_\tau^2} w_\tau(s) v_1^{\mathrm{APT}}(s) d\, s, \qquad (34)$$

where

$$w_\tau(s) = \frac{1}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right),$$

V_{ud} and S_{EW} denote the flavor mixing matrix element and an electro-weak correction term respectively. The condition $R_{\tau,V}^{\text{pert.}}|_{s>s_\text{p}} = R_{\tau,V}^{\text{exp.}}|_{s>s_\text{p}}$ leads to the equation

$$\int_{s_\text{p}}^{m_\tau^2} w_\tau(s) v_1^{\text{APT}}(s) d\, s = \int_{s_\text{p}}^{m_\tau^2} w_\tau(s) v_1^{\text{exp.}}(s) d\, s. \qquad (35)$$

Using relation (25), we express the left hand side of (35) in terms of the effective spectral density. By integrating by parts, after some algebra, we obtain

$$\int_{s_{\rm p}}^{m_\tau^2} w_\tau(s) v_1^{\rm APT}(s) d\,s = \frac{1}{4}\left(1-\frac{s_{\rm p}}{m_\tau^2}\right)^3\left(1+\frac{s_{\rm p}}{m_\tau^2}\right)(1+r(s_{\rm p}))$$
$$-\frac{1}{4\pi}\int_{s_{\rm p}}^{m_\tau^2}\frac{\rho_{\rm eff}(s)}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)d\,s. \qquad (36)$$

Numerical Results for the Parameters

To extract the parameters $s_{\rm p}$ and Λ from the data we have to solve the system of equations

$$\Phi_1(s_{\rm p}, \Lambda^2) = \int_0^{s_{\rm p}} v_1^{\rm exp.}(s) d\,s, \qquad (37)$$

$$\Phi_2(s_{\rm p}, \Lambda^2) = \int_{s_{\rm p}}^{m_\tau^2} w_\tau(s) v_1^{\rm exp.}(s) d\,s, \qquad (38)$$

where

$$\Phi_1(s_{\rm p},\Lambda^2) = \frac{s_{\rm p}}{2}(1+r(s_{\rm p})) + \frac{1}{2\pi}\int_0^{s_{\rm p}} \rho_{\rm eff}(\sigma)d\,\sigma + \frac{c_{\rm L}}{2}\Lambda^2, \qquad (39)$$

$$\begin{aligned}\Phi_2(s_{\rm p},\Lambda^2) &= (1-\bar{s}_{\rm p})^3(1+\bar{s}_{\rm p})\frac{(1+r(s_{\rm p}))}{4} \\ &-\frac{1}{4\pi}\int_{s_{\rm p}}^{m_\tau^2} \frac{\rho_{\rm eff}(s)}{s}(1-\bar{s})^3(1+\bar{s})d\,s,\end{aligned} \qquad (40)$$

with $\bar{s}_{\rm p} = s_{\rm p}/m_\tau^2$ and $\bar{s} = s/m_\tau^2$. The right hand sides of Eqs. (37)-(38) are determined in terms of the empirical function $v_1^{\rm exp.}(s)$. We reconstruct the experimental vector spectral function from the ALEPH 2005 spectral data for the vector invariant mass squared distribution which is publicly available. To interpolate the spectral function between the measured (at discrete points) values, we use cubic splines.

We solve the system of equations (37)-(38) numerically using various approximations to the Adler function.

We also determine the experimental uncertainties on the parameters coming from the uncertainties of the vector invariant mass squared distribution. The correlations between the errors of the distribution are properly taken into account using covariance matrixes.

Let us test convergence of the numerical results in perturbation theory. We use consecutive approximations to the Adler function from LO to N^4LO. For the unknown $\mathcal{O}(\alpha_s^5)$ correction, we use the geometric estimate $d_5 = d_4(d_4/d_3) = 378 \pm 378$. The results for the extracted values of the parameters are presented in Table

Observable	Approximation to the Adler function				
	LO	NLO	N^2LO	N^3LO	N^4LO
$s_{\rm p}$ GeV^2	1.7069	1.7098	1.7087	1.7069	1.7053
Λ GeV	0.4864	0.3778	0.3483	0.3316	0.3225
$\alpha_s(m_\tau^2)$	0.4010	0.3373	0.3214	0.3126	0.3078

Tab. 3

Formally, we may write a series for the numerical value of the coupling constant as follows

$$\alpha_s(m_\tau^2)|_{\rm N^4LO} = \alpha_s(m_\tau^2)|_{\rm LO} + \sum_{k=1}^{4} \Delta_k,$$

where $\Delta_k = \alpha_s(m_\tau^2)|_{\rm N^kLO} - \alpha_s(m_\tau^2)|_{\rm N^{k-1}LO}$. Using the numbers listed in Table 3 (we use abbreviation CI$^+$ for the modified CIPT accepted in this paper) we obtain the series

$$\alpha_s(m_\tau^2)|^{\rm CI^+}_{\rm N^4LO} = 0.4010 - 0.0638 - 0.0159 - 0.0088 - 0.0047. \quad (41)$$

The changes of the leading term induced by the consecutive corrections in the series are found to be: 15.9%, 4.0%, 2.2% and 1.2%. Using the standard CIPT to analyze the same ALEPH data we obtain (we use for PT correction to the τ decay rate the value $\delta^{(0)}_{\text{exp.}} = 0.2091 \pm 0.0065_{\text{exp.}}$ extracted from ALEPH data)

$$\alpha_s(m_\tau^2)|^{\text{CIPT}}_{\text{N}^4\text{LO}} = 0.485 - 0.095 - 0.023 - 0.013 - 0.007. \quad (42)$$

We see that within CIPT the corrections provide slightly larger changes of the leading term: 19.6%, 4.7%, 2.7% and 1.4%. One finds that $\Delta_k(\text{CIPT})/\Delta_{\text{k}}(\text{CI}^+) \approx 1.5$ for $k = 1-4$. So that the new series converges more rapidly than the series standard one.

The above series is used to find the so called indicative estimate of the theoretical uncertainty: it is determined as a half of the last retained term in the series

The error defined in this way is heuristic and indicative. From the series (41), we obtain the estimates within CI$^+$

$$\begin{aligned}
\alpha_s(m_\tau^2)|_{\mathrm{NLO}} &= 0.3373 \pm 0.0160_{\mathrm{exp.}} \pm 0.0319_{\mathrm{th}} \\
\alpha_s(m_\tau^2)|_{\mathrm{N^2LO}} &= 0.3214 \pm 0.0158_{\mathrm{exp.}} \pm 0.0079_{\mathrm{th}} \\
\alpha_s(m_\tau^2)|_{\mathrm{N^3LO}} &= 0.3126 \pm 0.0145_{\mathrm{exp.}} \pm 0.0044_{\mathrm{th}} \\
\alpha_s(m_\tau^2)|_{\mathrm{N^4LO}} &= 0.3078 \pm 0.0138_{\mathrm{exp.}} \pm 0.0024_{\mathrm{th}}, \qquad (43)
\end{aligned}$$

here we have also included the experimental errors. Analogically, from the CIPT series (42), one obtains

$$\begin{aligned}
\alpha_s(m_\tau^2)|_{\mathrm{NLO}} &= 0.3904 \pm 0.0109_{\mathrm{exp.}} \pm 0.0475_{\mathrm{th}} \\
\alpha_s(m_\tau^2)|_{\mathrm{N^2LO}} &= 0.3669 \pm 0.0093_{\mathrm{exp.}} \pm 0.0118_{\mathrm{th}} \\
\alpha_s(m_\tau^2)|_{\mathrm{N^3LO}} &= 0.3537 \pm 0.0084_{\mathrm{exp.}} \pm 0.0066_{\mathrm{th}} \\
\alpha_s(m_\tau^2)|_{\mathrm{N^4LO}} &= 0.3470 \pm 0.0081_{\mathrm{exp.}} \pm 0.0034_{\mathrm{th}}, \qquad (44)
\end{aligned}$$

The N^4LO estimates in (43) and (44) correspond to the central value $d_5 = 378$. The additional theoretical error in the coupling constant induced from the uncertainty in the fifth order unknown coefficient ($d_5 = 378 \pm 378$) takes the values 0.0045 ($\approx 1.5\%$) and 0.0065 ($\approx 1.9\%$) in the new and standard extraction procedures respectively. We see that the indicative estimates of the theoretical error within the new procedure are smaller than in standard one. In contrast to this, the experimental errors on the values of α_s within the new procedure increases by the factor of 1.76. It is remarkable that a more reliable estimate of the theoretical error presented in the literature for the case CIPT is close to the N^3LO and N^4LO values of the indicative error given in formula (44).

Similarly, determining the indicative theoretical errors on the parameter $s_{\rm p}$, we find stable results

$$
\begin{aligned}
s_{\rm p}|_{\rm NLO} &= 1.7098 \pm 0.0544_{\rm exp} \pm 0.0015_{\rm th} \\
s_{\rm p}|_{\rm N^2LO} &= 1.7087 \pm 0.0539_{\rm exp} \pm 0.0006_{\rm th} \\
s_{\rm p}|_{\rm N^3LO} &= 1.7069 \pm 0.0539_{\rm exp} \pm 0.0009_{\rm th} \\
s_{\rm p}|_{\rm N^4LO} &= 1.7053 \pm 0.0539_{\rm exp} \pm 0.0008_{\rm th}. \qquad (45)
\end{aligned}
$$

It is reasonable to investigate the applicability of the perturbation theory in the CI^+ framework, since this expansion formally depends on the small energy scale $\sqrt{s_{\rm p}} \approx 1.31$GeV. The issue of the applicability of perturbation theory in τ decays has been previously addressed in the literature

M. Girone and M. Neubert, (1996)

It is desirable to investigate numerically the convergence of the perturbative expansion within CI$^+$. Let us consider the expression for the τ-lepton decay rate into hadrons of invariant mass $s > s_{\rm p}$. This rate within CI$^+$ is approximated by a non-power series.

$$\hat{R}^{\rm pert.}_{\tau,{\rm V}}|_{s>s_{\rm p}} = R^{\rm pert.}_{\tau,{\rm V}}|_{s>s_{\rm p}}/\{6|V_{ud}|S_{EW}\} = \sum_{k=0}^{5} d_k \mathfrak{A}_k(m_\tau^2, s_{\rm p}) \quad (46)$$

where

$$\mathfrak{A}_0(m_\tau^2, s_{\rm p}) = f(s_{\rm p}/m_\tau^2), \quad (47)$$

$$\mathfrak{A}_{k\geq 1}(m_\tau^2, s_{\rm p}) = r_k(s_{\rm p}) f(s_{\rm p}/m_\tau^2) - \frac{1}{\pi}\int_{s_{\rm p}}^{m_\tau^2} \frac{f(\sigma/m_\tau^2)}{\sigma} \rho_k(\sigma)\, d\sigma, \quad (48)$$

here we have used the notations: $f(x) = \frac{1}{4}(1-x)^3(1+x)$ and

$$\begin{aligned} \rho_k(\sigma) &= \text{Im}\{\text{a}_\text{s}^\text{k}(-\sigma - \imath 0)\}, \\ r_k(s_\text{p}) &= \frac{1}{\pi}\int_{s_\text{p}}^{\infty} \frac{\rho_k(\sigma)}{\sigma} d\,\sigma. \end{aligned} \tag{49}$$

The first term in the series (46), $\mathfrak{A}_0$, corresponds to the (modified) parton level contribution to the rate. We calculate the functions $\mathfrak{A}_k$ numerically by using analytic expressions for the functions $\rho_k(\sigma)$ In the calculation, we employ the four-loop running coupling. For the parameters s_p and $\Lambda \equiv \Lambda_{\overline{\text{MS}}}$, we use the values extracted from the ALEPH data within APT$^+$ at N^{4}LO. Using analytically known coefficients d_k, $k = 0-4$ and the estimate $d_5 = 378$, we obtain from Eq. (46) the expansion

$$\begin{aligned} \widehat{R}_{\tau,\text{V}}^{\text{pert.}}|_{s>s_\text{p}} &= 0.3747 \cdot 10^{-1} + 0.3275 \cdot 10^{-2} + 0.3937 \cdot 10^{-3} \\ &+ 0.9270 \cdot 10^{-4} + 0.3304 \cdot 10^{-4} + (0.6047 \cdot 10^{-5}) \approx 0.04127. \end{aligned} \tag{50}$$

Consider now the non-power expansion for the perturbation theory correction $\delta^{(0)}$ obtained within CIPT

$$\delta^{(0)}_{\mathrm{CI}} = \sum_{k=1} d_k \mathcal{A}_k(m_\tau^2), \tag{51}$$

where

$$\mathcal{A}_k(m_\tau^2) = \frac{1}{\pi}\int_0^\pi \mathrm{Re}\{(1-e^{\imath\phi})(1+e^{\imath\phi})^3 a_s^k(m_\tau^2 e^{\imath\phi})\} d\,\phi,$$

to calculate these functions numerically, we employ the numerical value for the scale parameter Λ which is extracted from the ALEPH data within CIPT at N^{4}LO (see Table). At N^{4}LO, the expansion (51) can be rewritten as

$$\delta^{(0)}_{\mathrm{CI}} = 0.1513 + 0.3081\cdot 10^{-1} + 0.1276\cdot 10^{-1} + 0.9012\cdot 10^{-2} + (0.5233\cdot 10^{-2}) \tag{52}$$

Comparing the numerical expansions in Eqs. (50) and (52), one sees that the CI$^+$ series (50) displays a faster convergence. In the CIPT expansion (52), the corrections provide a 38% change of the leading term. In contrast, in the CI^{+} expansion (50) they provide only a 16% change of the leading term (we recall that the leading QCD correction in (50) is the second term in the series).

● It is convenient to perform evolution of the α_s results to the reference scale $M_z = 91.187\,\mathrm{GeV}$. This is done by using RG equation and appropriate matching conditions at the heavy quark (charm and bottom) thresholds We perform the matching at the matching scale $m_{\mathrm{th}} = 2\mu_{\mathrm{h}}$ where μ_{h} is a scale invariant $\overline{\mathrm{MS}}$ mass of the heavy quark $\mu_{\mathrm{h}} = \overline{m}_{\mathrm{h}}(\mu_{\mathrm{h}})$. We assume for the scale invariant $\overline{\mathrm{MS}}$ masses the values $\mu_{\mathrm{c}} = 1.27^{+0.07}_{-0.11}\,\mathrm{GeV}$ and $\mu_{\mathrm{b}} = 4.20^{+0.17}_{-0.07}\,\mathrm{GeV}$ In the evolution procedure, we have used the exact numeric four-loop running coupling. In Table, we compare the estimates for $\alpha_s(M_z^2)$ obtained within the new CI$^+$ and standard CIPT procedures.

Approximation	$\alpha_s(M_z^2)\|_{\mathrm{CI}^+}$	$\alpha_s(M_z^2)\|_{\mathrm{CIPT}}$
N^{2}LO	$0.1187 \pm 0.0019 \pm 0.0005$	$0.1238 \pm 0.0009 \pm 0.0005$
N^{3}LO	$0.1176 \pm 0.0018 \pm 0.0005$	$0.1224 \pm 0.0009 \pm 0.0005$
N^{4}LO	$0.1170 \pm 0.0018 \pm 0.0005$	$0.1217 \pm 0.0009 \pm 0.0005$

The lattice result

$$\alpha_s(M_z^2) = 0.1170 \pm 0.0012 \quad (\text{lattice Q. Masson et al.}). \tag{53}$$

Conclusion

❋ We have suggested a dispersive approach to study the hadronic decay rate of the τ lepton into non-strange vector channel.

❋ In the low energy region the spectral function is determined from the experiment. In the range $s > s_{\rm p}$, where $s_{\rm p} > 0$ is the onset of perturbative QCD, the spectral function is approximated within APT. Thus, the quark-hadron duality is used in the limited range $s_{\rm p} < s < m_\tau^2$.

❋ We have assumed that the dominant non-perturbative contributions to the spectral function comes from the low energy region $0 < s < s_{\rm p}$. Then the new approach enabled us to avoid application of the Wilson OPE in the Minkowski region, the main source of possible Duality Violations. On the other hand, in Euclidean region the OPE is accepted.

❋ We derive the system of equations for the strong coupling constant $\alpha_s(m_\tau^2)$ and the energy squared $s_{\rm p}$, the onset of perturbative QCD. According to this equations the parameters are determined as functionals of the experimental spectral function, the latter is constructed from the ALEPH data using cubic splines. Solving the system numerically we have extracted numerical values of the strong coupling constant $\alpha_s(m_\tau^2)$ and the energy squared $s_{\rm p}$. The errors on the parameters coming from the errors on the invariant mass distributions are correctly determined.

❋ In the calculations, we have used the series Lambert-W solution to the RG equation for the $\overline{\rm MS}$ running coupling to the four loop order. This solution enabled us to construct analytic formulas for calculation the effective spectral density, the central object for the perturbative calculations. For the QCD correction to the Adler function we have used the perturbative approximations up to N4LO.

❋ We have compared the values for strong coupling constant $\alpha_s(m_\tau^2)$ extracted within new approach (CI^+) and CIPT order by order in perturbation theory. It is remarkable that the central values of the coupling constant extracted within CI^+ in different orders of perturbation theory become systematically smaller as compared to the corresponding values obtained within CIPT (cf. formulas (43) and (44)). The changes in the central values are not within the quoted experimental and theoretical errors. At N^3LO The central values of $\alpha_s(m_\tau^2)$ in formulas (43) and (44) differ from each other in about 2.7 standard deviation, if the error is determined within CI^+, $\sigma = \sqrt{\sigma^2_{\text{exp.}} + \sigma^2_{\text{th.}}} \approx 0.0151$. Using the error obtained within CIPT, $\sigma \approx 0.0107$, one finds even large difference, $3.8\,\sigma$ [a].

❋ We have examined numerically the convergence of the perturbative series for the τ decay rate within the new approach and CIPT. We have confirmed that the series within CI^+ exhibits more fast convergence.

❋ The new procedure is based on the Adler function which obey correct behavior in the low energy region, the task un-achievable within APT, CIPT or FOPT.

[a] Due to the larger experimental error obtained within CI^+, $\sigma_{\mathrm{CI}^+}/\sigma_{\mathrm{CIPT}} \approx 1.4$.

❋ At NNNNLO we have reproduced the Lattice result for the central value of the strong coupling constant.

Thanks for attention